A-Level Pure Mathematics

Core 1

Core 1 is a **NON CALCULATOR** module

Topic N° 2

Let *a* be for algebra

A-Level Pure Mathematics

A L G E B R A

Quadratic Equations & Inequalities

Core 1 is a **NON CALCULATOR** module

Chapter 1

Algebra : Core 1

1.1 Revision of GCSE Algebra

Question 1.

GCSE Examination Question from November 2008, 3H, Q2

(a) Factorise

$$7p - 21$$

[1 mark]

(b) Solve

$$4\ (x + 5) = 12$$

You must show sufficient working.

[3 marks]

Question 2.

GCSE Examination Question from June 2011, 3H, Q5

Show that

$$\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$$

[2 marks]

Question 3.

GCSE Examination Question from November 2008, 3H, Q8

(a) Simplify

(i)

$$p^5 \times p$$

(ii)

$$\frac{q^5}{q^3}$$

[2 marks]

(b) Expand and simplify

$$3\,(4x - 1) - 4\,(2x - 3)$$

[2 marks]

(c) Expand and simplify

$$(y + 3)\,(y + 5)$$

[2 marks]

Question 4.
GCSE Examination Question from May 2012, 3H, Q3

(a) Write as a single power of 2,

$$2^3 \times 2^6$$

[1 mark]

(b) Write as a single power of 3,

$$\frac{3^9}{3^4}$$

[1 mark]

(c)

$$\frac{5^n}{5^4 \times 5^6} = 5^3$$

Find the value of *n*.

[2 marks]

Question 5.
GCSE Examination Question from November 2008, 3H, Q14

(a) Factorise completely

$$9ab - 12b^2$$

[2 marks]

(b) Simplify

$$(2ab^2)^3$$

[2 marks]

Question 6.

GCSE Examination Question from May 2012, 3H, Q17

(a) Simplify

$$(3a^2b)^4$$

[2 marks]

(b) Simplify

$$(9c^8)^{\frac{1}{2}}$$

[2 marks]

Question 7.

(a) Factorise $x^2 - 8x + 12$

[2 marks]

(b) Factorise $x^2 - 81$

[1 mark]

Question 8.

Simplify the following algebraic expressions by first factorising the quadratics:

$$\frac{x^2 + 3x - 88}{x^2 - 2x - 48}$$

[2 marks]

Question 9.
Simplify fully

$$\frac{x^2 + 5x}{x^2 - 25}$$

[3 marks]

Question 10.
Express as a single fraction
(i)

$$\frac{2(2x + 7)}{5} + \frac{5(4x + 1)}{3}$$

[3 marks]

(ii)

$$\frac{2(5x + 4)}{3} - \frac{3(3x - 2)}{4}$$

[3 marks]

Question 11.

Simplify the following expression;

$$\frac{7}{(x + 4)} + \frac{5}{(x + 3)}$$

[3 marks]

Question 12.

Beginning "LHS = " show that;

$$\frac{3}{(x + 6)} + \frac{5}{(x - 10)} = \frac{8x}{(x + 6)(x - 10)}$$

[3 marks]

Question 13.

(a) Simplify

$$\frac{x^2}{x^2 - 5x}$$

[2 marks]

(b) Simplify

$$\frac{6}{2x - 9} - \frac{2}{2x + 3}$$

[4 marks]

Question 14.

Solve

$$\frac{x - 3}{2} + \frac{x - 5}{3} = 6$$

[4 marks]

Question 15.
Solve the equation;

$$\frac{x}{2} = \frac{2(x - 2)}{7}$$

[3 marks]

Question 16.
Find the two solutions to the equation;

$$\frac{x}{x + 3} = \frac{10}{x - 3}$$

[4 marks]

1.2 Answers (Revision of GCSE Algebra)

Answer 1.

(a) $7(p - 3)$

(b) -2

Answer 2.

$$\begin{aligned} LHS &= \frac{5}{6} - \frac{3}{4} \qquad\qquad LCM\{6,\ 4\} = 12 \\ &= \frac{2}{2} \times \frac{5}{6} - \frac{3}{4} \times \frac{3}{3} \\ &= \frac{10}{12} - \frac{9}{12} \\ &= \frac{1}{12} \\ &= RHS \qquad\qquad \square \end{aligned}$$

Answer 3.

(a) **(i)** p^6

(ii) q^2

(b) $4x + 9$

(c) $y^2 + 8y + 15$

Answer 4.

(a) 2^9

(b) 3^5

(c) $n = 13$

Answer 5.

(a) $3b(3a - 4b)$

(b) $8a^3b^6$

Answer 6.

(a) $81a^8b^4$

(b) $3c^4$

Answer 7.

(a) $(x - 2)(x - 6)$

(b) $(x + 9)(x - 9)$

Answer 8.

$$\frac{x + 11}{x + 6}$$

Answer 9.

$$\frac{x}{x - 5}$$

Answer 10.

(i)

$$\frac{112x + 67}{15}$$

(ii)

$$\frac{13x + 50}{12}$$

Answer 11.

$$\frac{12x + 41}{(x + 4)(x + 3)}$$

Answer 12.

$$\begin{aligned} LHS &= \frac{3}{(x + 6)} + \frac{5}{(x - 10)} \\ &= \frac{(x - 10)}{(x - 10)} \times \frac{3}{(x + 6)} + \frac{5}{(x - 10)} \times \frac{(x + 6)}{(x + 6)} \\ &= \frac{3x - 30 + 5x + 30}{(x + 6)(x - 10)} \\ &= \frac{8x}{(x + 6)(x - 10)} \\ &= RHS \qquad \square \end{aligned}$$

Answer 13.

(a)

$$\frac{x}{x - 5}$$

(b)

$$\frac{8x + 36}{(2x + 3)(2x - 9)}$$

Answer 14.

$x = 11$

Answer 15.

$x = -\frac{8}{3} \quad (-2.67)$

Answer 16.

$x = -2, \ 15$

Chapter 2

Algebra : Core 1

2.1 More Revision of GCSE Algebra (Homework)

Question 1.

GCSE Examination Question from November 2007, 4H, Q2

(a) Factorise

$$5x - 20$$

[1 mark]

(b) Factorise

$$y^2 + 6y$$

[2 marks]

Question 2.

GCSE Examination Question from May 2007, 3H, Q9

(a) Solve

$$5x - 4 = 2x + 7$$

[2 marks]

(b) Solve

$$\frac{7 - 2y}{4} = 2y + 3$$

[4 marks]

Question 3.

GCSE Examination Question from January 2013, 4H, Q8

(a) Factorise

$$n^2 + 8n$$

[2 marks]

(b) Expand and simplify

$$3(2x - 5) - 4(x + 3)$$

[2 marks]

(c) Expand and simplify

$$(y + 7)(y + 2)$$

[2 marks]

Question 4.

GCSE Examination Question from May 2008, 4H, Q6

Show that

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

[2 marks]

Question 5.

GCSE Examination Question from May 2007, 3H, Q5

(a) Simplify, leaving your answers in index form

(i) $7^5 \times 7^3$

(ii) $5^9 \div 5^3$

[2 marks]

(b) Solve

$$\frac{2^9 \times 2^4}{2^n} = 2^8$$

[2 marks]

Question 6.

GCSE Examination Question from May 2008, 3H, Q14

(a) Factorise

$$10y - 15$$

[1 mark]

(b) Factorise completely

$$9p^2q + 12pq^2$$

[2 marks]

(c) **(i)** Factorise

$$x^2 + 6x - 16$$

(ii) Solve

$$x^2 + 6x - 16 = 0$$

[3 marks]

Question 7.

GCSE Examination Question from January 2013, 4H, Q15

(a) Simplify

$$\frac{5x^5y^6}{x^2y^4}$$

[2 marks]

(b) Simplify

$$(2n^4)^3$$

[2 marks]

Question 8.

GCSE Examination Question from May 2006, 3H, Q13 (a) (c)

(a) Expand ans simplify

$$(3x - 5)(4x + 7)$$

[2 marks]

(b) Simplify

$$(64y^6)^{\frac{2}{3}}$$

[2 marks]

Question 9.

Simplify the following algebraic expressions by first factorising the quadratics:

$$\frac{x^2 + 2x - 24}{x^2 - 3x - 54}$$

[2 marks]

Question 10.

GCSE Examination Question from May 2004, 3H, Q16

Express this algebraic fraction as simply as possible

$$\frac{2x^2 - 3x - 20}{x^2 - 16}$$

[3 marks]

Question 11.

Express as a single fraction

(i)

$$\frac{(3x + 7)}{5} + \frac{(7x - 4)}{3}$$

[3 marks]

(ii)

$$\frac{5(4x + 1)}{2} - \frac{3(7x - 2)}{5}$$

[3 marks]

Question 12.

Simplify the following expression;

$$\frac{1}{(x+4)} + \frac{4}{(x+5)}$$

[3 marks]

Question 13.

Beginning "LHS = " show that;

$$\frac{8}{(x-4)} + \frac{2}{(x+6)} = \frac{10(x+4)}{(x-4)(x+6)}$$

[3 marks]

Question 14.

GCSE Examination Question from May 2006, 4H, Q4

Arul had x sweets.

Nikos had four times as many sweets as Arul.

(a) Write down an expression, in terms of x, for the number of sweets Nikos had.

[1 mark]

Nikos gave 6 of his sweets to Arul.

Now they both have the same number of sweets.

(b) Use this information to form an equation in x.

[2 marks]

(c) Solve your equation to find the number of sweets that Arul had at the start.

[2 marks]

Question 15.

(a) Simplify

$$\frac{x^2 - 7x}{x^2}$$

[2 marks]

(b) Simplify

$$\frac{2}{x + 1} - \frac{4}{2x + 3}$$

[4 marks]

Question 16.

Solve

$$\frac{x - 3}{2} + \frac{x - 2}{5} = 10$$

[4 marks]

Question 17.

GCSE Examination Question from January 2013, 4H, Q15

Solve

$$\frac{2}{5x - 2} = \frac{3}{6x + 1}$$

[4 marks]

Question 18.
Find the two solutions to the equation;

$$\frac{x}{x+5} = \frac{2}{x-7}$$

[4 marks]

2.2 Answers (More Revision of GCSE Algebra - Homework)

Answer 1.

(a) $5(x - 4)$

(b) $y(y + 6)$

Answer 2.

(a) $\frac{11}{3}$

(b) -0.5

Answer 3.

(a) $n(n + 8)$

(b) $2x - 27$

(c) $y^2 + 9y + 14$

Answer 4.

$$LHS = \frac{2}{3} + \frac{1}{4} \qquad LCM\{3, 4\} = 12$$

$$= \frac{4}{4} \times \frac{2}{3} + \frac{1}{4} \times \frac{3}{3}$$

$$= \frac{8}{12} + \frac{3}{12}$$

$$= \frac{11}{12}$$

$$= RHS \qquad \square$$

Answer 5.

(a) **(i)** 7^8

(ii) 5^6

(b) $n = 5$

Answer 6.

(a) $5(2y - 3)$

(b) $3pq(3p + 4q)$

(c) **(i)** $(x + 8)(x - 2)$

(ii) $x = -8$ or $x = 2$

Answer 7.

(a) $5x^3y^2$

(b) $8n^{12}$

Answer 8.

(a) $12x^2 + x - 35$

(b) $16y^4$

Answer 9.

$$\frac{x - 4}{x - 9}$$

Answer 10.

$$\frac{2x + 5}{x + 4}$$

Answer 11.

(i)

$$\frac{44x + 1}{15}$$

(ii)

$$\frac{58x + 37}{10}$$

Answer 12.

$$\frac{5x + 21}{(x + 5)(x + 4)}$$

Answer 13.

$$\begin{aligned} LHS &= \frac{8}{(x - 6)} + \frac{2}{(x + 6)} \\ &= \frac{(x + 6)}{(x + 6)} \times \frac{8}{(x - 4)} + \frac{2}{(x + 6)} \times \frac{(x - 4)}{(x - 4)} \\ &= \frac{8x + 48 + 2x - 8}{(x + 6)(x - 4)} \\ &= \frac{10x + 40}{(x - 4)(x + 6)} \\ &= \frac{10(x + 4)}{(x - 4)(x + 6)} \\ &= RHS \end{aligned}$$

□

Answer 14.

(a) $4x$

(b) $x + 6 = 4x - 6$

(c) $x = 4$

Answer 15.

(a)

$$\frac{x - 7}{x}$$

(b)

$$\frac{2}{(2x + 3)(x + 1)}$$

Answer 16.

$x = 17$

Answer 17.

$x = 2\frac{2}{3}$

Answer 18.

$x = -1, \; 10$

Chapter 3

Algebra : Core 1

3.1 Completing the square.

Example

Solve this equation by completing the square, giving an exact answer.

$$x^2 - 6x = 2$$

3.2 Exercise.

Question 1.

Solve these equations by completing the square, giving exact answers.

(i)

$$x^2 - 8x = 1$$

(ii)

$$x^2 + 2x = 5$$

(iii)

$$x^2 - 12x = 5$$

(iv)

$$x^2 + 14x + 30 = 0$$

Question 2.
By completing the square on

$$x^2 + 6x = 11$$

Show that the equation has solutions of the form $a \pm b\sqrt{c}$
where a, b and c are integers and c is square free.

Question 3.
Solve these equations by completing the square, giving exact answers.

(i)

$$(x - 5)(x + 3) = 1$$

(ii)

$$x + \frac{1}{x} = 4$$

(iii)

$$x(x + 4) = 7$$

(iv)

$$\frac{1}{(x + 1)} + x = 3$$

Question 4.
The following equation is to be solved by the method of completing the square;

$$\frac{2}{x} + \frac{17}{x^2} = 1$$

Show that the exact solutions are of the form;

$$x = a + b\sqrt{c}$$

where *a*, *b* and *c* are integers and *c* is square free.

Question 5.
The following equation is to be solved by the method of completing the square;

$$\frac{5}{2x} = \frac{x - 20}{4}$$

Show that the exact solutions are of the form;

$$x = a + b\sqrt{c}$$

where *a*, *b* and *c* are integers and *c* is square free.

Examination question from May 2005, Q3.

Question 6.

$$x^2 - 8x - 29 \equiv (x + a)^2 + b$$

where a and b are constants.

(a) Find the value of a and the value of b.

[3 marks]

(b) Hence, or otherwise, show that the roots of

$$x^2 - 8x - 29 = 0$$

are $c \pm d\sqrt{5}$ where c and d are integers to be found.

[3 marks]

Question 7.

$$x^2 - 3x + 1 \equiv (x + a)^2 + b$$

where a and b are rational constants.

(a) Find the value of a and the value of b.

(b) Hence, or otherwise, show that the roots of

$$x^2 - 3x + 1 = 0$$

are $c \pm d\sqrt{5}$ where c and d are rational constants to be found.

Question 8.
Find, as surds, the roots of the equation:

$$(x - 2)^2 = 2(x + 1)(x - 4)$$

Question 9.
The following equation is to be solved by the method of completing the square;

$$x^2 - 3x - 1 = 0$$

Show that the exact solutions are of the form;

$$x = \frac{a \pm \sqrt{b}}{2}$$

clearly stating the values of the integers, *a* and *b*.

Question 10.
Use the method of completing the square to solve the equation;

$$x - 2x^{\frac{1}{2}} - 1 = 0$$

Give your answers in the exact form;

$$x = a + b\sqrt{c}$$

where a, b and c are integers and c is square free.

3.3 Answers.

3.3.1 Solutions (3.1 Introductory Examples)

$$
\begin{aligned}
x^2 - 6x &= 2 \\
(x - 3)^2 - 9 &= 2 \\
(x - 3)^2 &= 11 \\
x - 3 &= \pm\sqrt{11} \\
x &= 3 \pm \sqrt{11}
\end{aligned}
$$

3.3.2 Solutions (3.2 Exercise)

Answer 1.

(i) $4 \pm \sqrt{17}$

(ii) $-1 \pm \sqrt{6}$

(iii) $6 \pm \sqrt{41}$

(iv) $-7 \pm \sqrt{19}$

Answer 2.

$-3 \pm 2\sqrt{5}$

Answer 3.

(i) $1 \pm \sqrt{17}$

(ii) $2 \pm \sqrt{3}$

(iii) $-2 \pm \sqrt{11}$

(iv) $1 \pm \sqrt{3}$

Answer 4.

$1 \pm 3\sqrt{2}$

Answer 5.

$10 \pm \sqrt{110}$

Answer 6.

(a) $a = -4$, $b = -45$

(b) $x = 4 \pm 3\sqrt{5}$, $c = 4$, $d = 3$

Answer 7.

(a) $a = -\frac{3}{2}$, $b = -\frac{5}{4}$

(b) $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$ $c = \frac{3}{2}$, $d = \frac{1}{2}$

Answer 8.

$x = 1 \pm \sqrt{13}$

Answer 9.

$x = \dfrac{3 \pm \sqrt{13}}{2}$ $a = 3$, $b = 13$

Answer 10.

$x = 3 \pm 2\sqrt{2}$

4.1 Completing The Square : Tougher Problems

4.1.1 Example

By completing the square, solve the equation :

$$2x^2 + 3x - 3 = 0$$

Give your answer in the form $a \pm b\sqrt{33}$ for some constants a and b.

4.1.2 Model Solution :

Pull out the coefficient of x^2 from the trems in x^2 and x :

$$2\,[\,x^2 + \frac{3}{2}x\,] - 3 = 0$$

Complete the square within the square brackets :

$$2\,[\,(x + \frac{3}{4})^2 - \frac{9}{16}\,] - 3 = 0$$

Expand the square brackets :

$$2\,(x + \frac{3}{4})^2 - \frac{9}{8} - 3 = 0$$

Tidy up :

$$2\,(x + \frac{3}{4})^2 - \frac{9}{8} - \frac{24}{8} = 0$$

$$2\,(x + \frac{3}{4})^2 - \frac{33}{8} = 0$$

$$(x + \frac{3}{4})^2 = \frac{33}{16}$$

Square root both sides :

$$x + \frac{3}{4} = \pm\sqrt{\frac{33}{16}}$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{33}}{4}$$

4.2 Exercise

Question 1.

Without using a calculator, use the method of completing the square to solve these equations;

(i) $2x^2 + 7x + 1 = 0$

(ii) $5x^2 + 7x + 1 = 0$

(iii) $3x^2 + 5x + 1 = 0$

(iv) $10x^2 + 3x - 2 = 0$

Question 2.
Without using a calculator, by completing the square, solve the equation;
$5x^2 + 4x - 2 = 0$

Question 3.
Without using a calculator, by completing the square, show that the solutions to the equation; $2x^2 - 12x + 17 = 0$ are;

$$x = 3 \pm \frac{\sqrt{2}}{2}$$

Question 4.

Without using a calculator, by completing the square, show that the solutions to the equation; $2x^2 + 2x - 1 = 0$ are;

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

Question 5.

By completing the square, solve the equation; $9x^2 + 6x - 17 = 0$

Question 6.
By completing the square, solve the equation; $9x^2 - 6x - 26 = 0$

Appendix : Completing the Square

How to solve quadratics

(Published in M500, The Open University Mathematics Magazine)

To solve the quadratic equation:

$$ax^2 + bx + c = 0$$

Multiply through by $4a$:

$$[\ 4a^2x^2 + 4abx\ ] + 4ac = 0$$

Complete the square within the square brackets:

$$[\ (2ax + b)^2 - b^2\ ] + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac$$

Square-root both sides:

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

And the well-known formula is there for the taking:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4.3 Answers (4.2 Exercise)

Answer 1.

(i)

$$2x^2 + 7x + 1 = 0$$

$$2\left[x^2 + \frac{7}{2}x\right] + 1 = 0$$

$$2\left[\left(x + \frac{7}{4}\right)^2 - \frac{49}{16}\right] + 1 = 0$$

$$2\left(x + \frac{7}{4}\right)^2 - \frac{49}{8} + \frac{8}{8} = 0$$

$$2\left(x + \frac{7}{4}\right)^2 - \frac{41}{8} = 0$$

$$2\left(x + \frac{7}{4}\right)^2 = \frac{41}{8}$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{41}{16}$$

$$x + \frac{7}{4} = \pm\sqrt{\frac{41}{16}}$$

$$x = -\frac{7}{4} \pm \frac{\sqrt{41}}{4}$$

(ii)

$$5x^2 + 7x + 1 = 0$$

$$5\left[x^2 + \frac{7}{5}x\right] + 1 = 0$$

$$5\left[\left(x + \frac{7}{10}\right)^2 - \frac{49}{100}\right] + 1 = 0$$

$$5\left(x + \frac{7}{10}\right)^2 - \frac{49}{20} + \frac{20}{20} = 0$$

$$5\left(x + \frac{7}{10}\right)^2 - \frac{29}{20} = 0$$

$$5\left(x + \frac{7}{10}\right)^2 = \frac{29}{20}$$

$$\left(x + \frac{7}{10}\right)^2 = \frac{29}{100}$$

$$x + \frac{7}{10} = \pm\sqrt{\frac{29}{100}}$$

$$x = -\frac{7}{10} \pm \frac{\sqrt{29}}{10}$$

(iii)

$$3x^2 + 5x + 1 = 0$$

$$3\,[\,x^2 + \frac{5}{3}x\,] + 1 = 0$$

$$3\,[\,(x + \frac{5}{6})^2 - \frac{25}{36}\,] + 1 = 0$$

$$3\,(\,x + \frac{5}{6})^2 - \frac{25}{12} + \frac{12}{12} = 0$$

$$3\,(\,x + \frac{5}{6})^2 - \frac{13}{12} = 0$$

$$3\,(\,x + \frac{5}{6})^2 = \frac{13}{12}$$

$$(\,x + \frac{5}{6})^2 = \frac{13}{36}$$

$$x + \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

$$x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

(iv)

$$10x^2 + 3x - 2 = 0$$

$$10\,[\,x^2 + \frac{3}{10}x\,] - 2 = 0$$

$$10\,[\,(\,x + \frac{3}{20})^2 - \frac{9}{400}\,] - 2 = 0$$

$$10\,(\,x + \frac{3}{20})^2 - \frac{9}{40} - \frac{80}{40} = 0$$

$$10\,(\,x + \frac{3}{20})^2 - \frac{89}{40} = 0$$

$$10\,(\,x + \frac{3}{20})^2 = \frac{89}{40}$$

$$(x + \frac{3}{20})^2 = \frac{89}{400}$$

$$x + \frac{3}{20} = \pm\sqrt{\frac{89}{400}}$$

$$x = -\frac{3}{20} \pm \frac{\sqrt{89}}{20}$$

Answer 2.

$$5x^2 + 4x - 2 = 0$$

$$5\left[x^2 + \frac{4}{5}x\right] - 2 = 0$$

$$5\left[\left(x + \frac{2}{5}\right)^2 - \frac{4}{25}\right] - 2 = 0$$

$$5\left(x + \frac{2}{5}\right)^2 - \frac{4}{5} - \frac{10}{5} = 0$$

$$5\left(x + \frac{2}{5}\right)^2 - \frac{14}{5} = 0$$

$$5\left(x + \frac{2}{5}\right)^2 = \frac{14}{5}$$

$$\left(x + \frac{2}{5}\right)^2 = \frac{14}{25}$$

$$x + \frac{2}{5} = \pm\sqrt{\frac{14}{25}}$$

$$x = -\frac{2}{5} \pm \frac{\sqrt{14}}{5}$$

Answer 3.

$$2x^2 - 12x + 17 = 0$$

$$2\left[x^2 - 6x\right] + 17 = 0$$

$$2\left[(x - 3)^2 - 9\right] + 17 = 0$$

$$2(x - 3)^2 - 18 + 17 = 0$$

$$2(x - 3)^2 - 1 = 0$$

$$2(x - 3)^2 = 1$$

$$(x - 3)^2 = \frac{1}{2}$$

$$x - 3 = \pm\sqrt{\frac{1}{2}}$$

$$x = 3 \pm \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$x = 3 \pm \frac{\sqrt{2}}{2}$$

Answer 4.

$$2x^2 + 2x - 1 = 0$$

$$2[x^2 + x] - 1 = 0$$

$$2[(x + \frac{1}{2})^2 - \frac{1}{4}] - 1 = 0$$

$$2(x + \frac{1}{2})^2 - \frac{1}{2} - \frac{2}{2} = 0$$

$$2(x + \frac{1}{2})^2 - \frac{3}{2} = 0$$

$$2(x + \frac{1}{2})^2 = \frac{3}{2}$$

$$(x + \frac{1}{2})^2 = \frac{3}{4}$$

$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}}$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

Answer 5.

$$9x^2 + 6x - 17 = 0$$

$$9[x^2 + \frac{2}{3}x] - 17 = 0$$

$$9[(x + \frac{1}{3})^2 - \frac{1}{9}] - 17 = 0$$

$$9(x + \frac{1}{3})^2 - 1 - 17 = 0$$

$$9(x + \frac{1}{3})^2 - 18 = 0$$

$$9(x + \frac{1}{3})^2 = 18$$

$$(x + \frac{1}{3})^2 = 2$$

$$x + \frac{1}{3} = \pm\sqrt{2}$$

$$x = -\frac{1}{3} \pm \sqrt{2}$$

Answer 6.

$$9x^2 - 6x - 26 = 0$$

$$9\left[x^2 - \frac{2}{3}x\right] - 26 = 0$$

$$9\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right] - 26 = 0$$

$$9\left(x - \frac{1}{3}\right)^2 - 1 - 26 = 0$$

$$9\left(x - \frac{1}{3}\right)^2 - 27 = 0$$

$$9\left(x - \frac{1}{3}\right)^2 = 27$$

$$\left(x - \frac{1}{3}\right)^2 = 3$$

$$x - \frac{1}{3} = \pm\sqrt{3}$$

$$x = \frac{1}{3} \pm \sqrt{3}$$

Chapter 5

Algebra : Core 1

5.1 Solving Quadratic Equations

A quadratic equation is of the form;

$$y = ax^2 + bx + c$$

In this equation x and y are variables, whereas a, b and c are constants.

The graph of this equation is a ***quadratic curve***, also called a ***parabola***. It's a useful shape, used in, for example, car headlights, electric heaters, radio telescopes and solar furnesses.

Often the mathematical interested is in where (if anywhere) a given quadratic curve crosses the x-axis. It does this when it has no height. That is, when y is zero. When we talk about ***solving a quadratic equation*** we mean finding the values of x (often called *roots*) for which $y = 0$.

i.e. Solving;

$$ax^2 + bx + c = 0$$

Such equations can be solved by

(i) 'Guessing the brackets' or factorisation.

(ii) Completing the square.

(iii) Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5.2 Example

Solve the equation

$$x^2 + 14x + 40 = 0$$

(i) By "guessing the brackets' or factorisation.

$$x^2 + 14x + 40 = 0$$

(ii) By completing the square.

$$x^2 + 14x + 40 = 0$$

(iii) Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 + 14x + 40 = 0$$

Here is the graph of the quadratic;

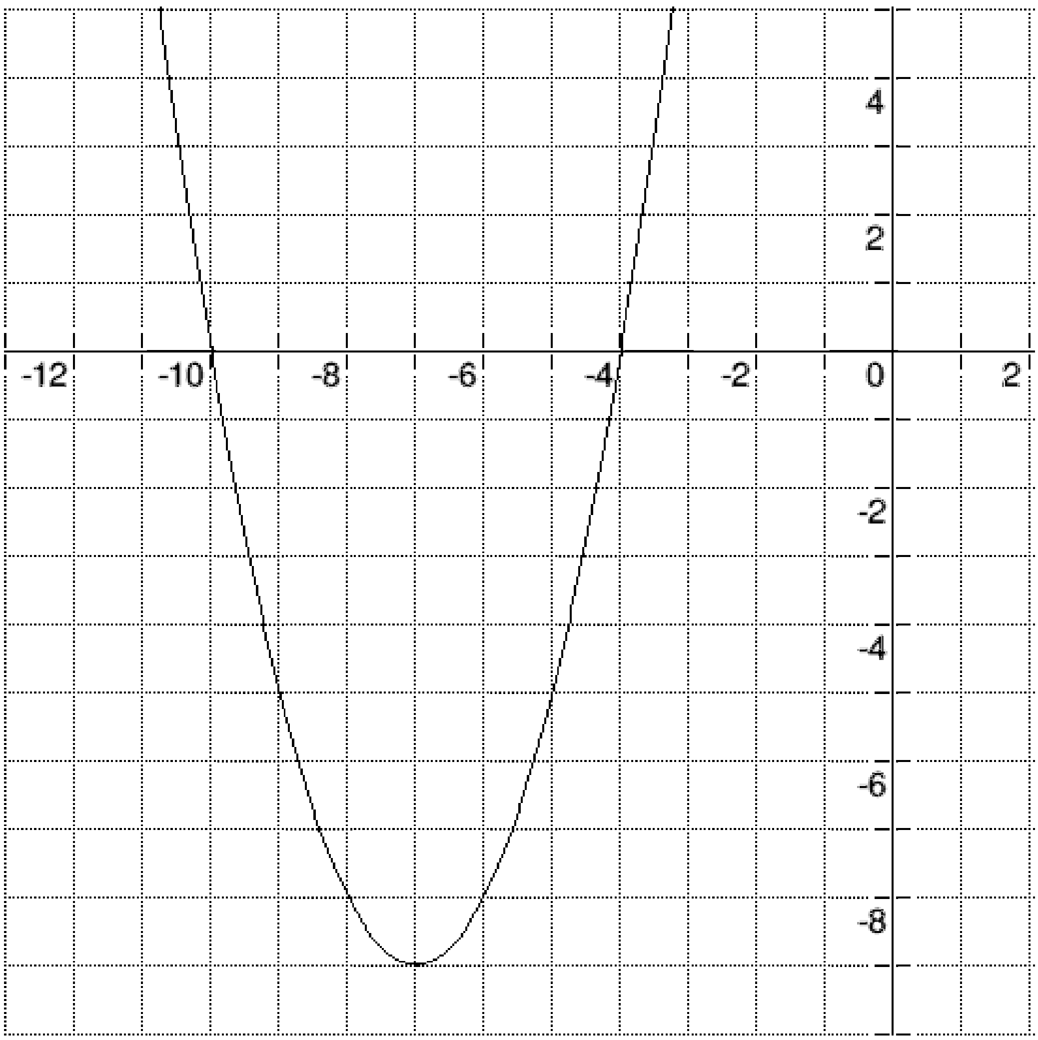
$$y = x^2 + 14x + 40$$

We solved:

$$x^2 + 14x + 40 = 0$$

And we got that:

$$x = -10, \ -4$$



5.3 Exercise

Question 1

Use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve the following quadratic equations.

Give solutions as exact expressions, involving square roots if necessary.

(i) $14x^2 + 11x + 2 = 0$

(ii) $7x^2 - 3x - 2 = 0$

(iii) $9x^2 - 16x + 7 = 0$

(iv) $12x^2 + 8x - 12 = 0$

Question 2.

Try to use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve the equation

$$x^2 + x + 1 = 0$$

What goes wrong ?

Question 3.

The part of the quadratic equation solving formula that sits under the square-root is called the discriminant, *D*.

$$D = b^2 - 4ac$$

If $D > 0$, the quadratic equation solving formula has two solutions.
If $D = 0$, there is one solution.
If $D < 0$, there are no solutions.

Calculate the value of the discriminant, *D*, and hence state the number of distinct real roots of each of these equations.
DO NOT SOLVE THE EQUATIONS !

(i)

$$x^2 + 4x - 5 = 0$$

(ii)

$$x^2 + 4x + 5 = 0$$

(iii)

$$2x^2 + x - 5 = 0$$

(iv)

$$x^2 + 4x + 4 = 0$$

(v)

$$3x^2 + 4x + 2 = 0$$

(vi)

$$4x^2 - 25 = 0$$

C1 Examination question from June 2009, Q6.

Question 4.

The equation $x^2 + 3px + p = 0$, where p is a non-zero constant, has equal roots.
Find the value of p.

5.4 Answers

5.4.1 Solutions to 5.2 Example

(i)

$$x^2 + 14x + 40 = 0$$

$$(x + 4)(x + 10) = 0$$

$$\textit{either} \quad x + 4 = 0 \qquad \textit{in which case} \quad x = -4$$

$$\textit{or} \quad x + 10 = 0 \qquad \textit{in which case} \quad x = -10$$

(ii)

$$x^2 + 14x + 40 = 0$$

$$[x^2 + 14x] + 40 = 0$$

$$[(x + 7)^2 - 49] + 40 = 0$$

$$(x + 7)^2 - 9 = 0$$

$$(x + 7)^2 = 9$$

$$x + 7 = \pm 3$$

$$x = -7 \pm 3$$

$$i.e. \; x = -10, \quad -4$$

(iii)

$$x^2 + 14x + 40 = 0$$

$$x = \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 40}}{2}$$

$$= \frac{-14 \pm \sqrt{196 - 160}}{2}$$

$$= \frac{-14 \pm \sqrt{36}}{2}$$

$$= \frac{-14 \pm 6}{2}$$

$$= -10, \quad -4$$

On the graph observe the x-axis crossing points are at (- 10, 0) and (- 4, 0)

Also, from the rewrite of the curve's equation

$$y = x^2 + 14x + 40$$

as

$$y = (x + 7)^2 - 9$$

it can be noticed that the minimum value is at (- 7, - 9)

5.4.2 Solutions to 5.3 Exercise

Answer 1.

(i)

$$-\frac{2}{7}, -\frac{1}{2}$$

(ii)

$$\frac{3 \pm \sqrt{65}}{14}$$

(iii)

$$1, \frac{7}{9}$$

(iv)

$$\frac{-1 \pm \sqrt{10}}{3}$$

Answer 2.

Formula yields an expression containing the square root of a negative number

i.e. $\dfrac{-1 \pm \sqrt{-3}}{2}$

Answer 3.

(i)	D = 36 which is +ve	$\therefore$ 2 solutions
(ii)	D = - 4 which is -ve	$\therefore$ 0 solutions
(iii)	D = 41 which is +ve	$\therefore$ 2 solutions
(iv)	D = 0	$\therefore$ 1 solutions
(v)	D = - 8 which is -ve	$\therefore$ 0 solutions
(vi)	D = 400 which is +ve	$\therefore$ 2 solutions

Answer 4.

$$p = \frac{4}{9}$$

Chapter 6

Algebra : Core 1

6.1 The Discriminant

We've been looking at solving quadratic equations, and have started to notice that the number of roots (solutions) is not always two.

The generalised quadratic;

$$a x^2 + b x + c = 0$$

has solutions given by;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The piece of formula under the square root sign determines the number of roots. This crucial piece of formula is termed the discriminant;

$$D = b^2 - 4ac$$

Here are three examples illustrating the dramatic effect the discriminant has on the number of roots:

$x^2 + x + 1 = 0$	$x^2 + 2x + 1 = 0$	$x^2 + x - 2 = 0$
$a = 1$	$a = 1$	$a = 1$
$b = 1$	$b = 2$	$b = 1$
$c = 1$	$c = 1$	$c = -2$
$x = \frac{-1 \pm \sqrt{-3}}{2}$	$x = \frac{-2 \pm \sqrt{0}}{2}$	$x = \frac{-1 \pm \sqrt{9}}{2}$
$x = \{ \}$	$x = \{-1\}$	$x = \{-2, 1\}$
No Roots	*One Root*	*Two Roots*
$D < 0$	$D = 0$	$D > 0$
D is Negative	*D is Zero*	*D is Positive*

Sometimes *One Root* is refered to as *a repeated root*.

This can all be used backwards so that, for example, if you are told that an equation has a repeated root then you know that;

$$D = 0$$

$$i.e. \quad b^2 - 4ac = 0$$

6.2 Exercise.

Question 1.

For each of the following equations,

(a) Calculate the value of the discriminant,

(b) State if the equation has 0, 1 or 2 roots.

DO NOT SOLVE THE EQUATIONS !

(i)

$$x^2 + 4x - 3 = 0$$

(ii)

$$x^2 - 2x + 8 = 0$$

(iii)

$$x^2 + 6x + 9 = 0$$

(iv)

$$(x + 4)(x + 3) + 1 = 0$$

(v)

$$\frac{(x - 2)}{(x + 1)} = x + 3$$

(vi)

$$9x^2 + 30x + 25 = 0$$

Question 2.
The following equation has no real roots

$$x^2 - 3x - k = 0$$

where k is some fixed constant.

Show that k must be less than - 2.25.

Question 3.
The following equation has two distinct real roots;

$$x^2 + kx + 4 = 0$$

where k is some fixed constant.

(i) Show that $k^2 > 16$.

(ii) Which of the following equations will have two distinct real roots ?

$$x^2 + 5x + 4 = 0$$

$$x^2 + 3x + 4 = 0$$

$$x^2 - x + 4 = 0$$

$$x^2 - 7x + 4 = 0$$

(iii) Solve by completing the square;

$$x^2 + 6x + 4 = 0$$

Question 4.
The following equation has a repeated root;

$$x^2 + (2k + 10)x + (k^2 + 5) = 0$$

where *k* is some fixed constant.

Determine the value of *k*.

C1 Examination question from January 2005, Q3.
Question 5.
Given that the equation

$$kx^2 + 12x + k = 0$$

where *k* is a positive constant, has equal roots, find the value of *k*.

[4 marks]

C1 Examination question from May 2006, Q8.

Question 6.

The equation

$$x^2 + 2px + (3p + 4) = 0$$

where p is a positive constant, has equal roots.

(a) Find the value of p.

[4 marks]

(b) For this value of p, solve the equation

$$x^2 + 2px + (3p + 4) = 0$$

[2 marks]

Question 7.

Explain why

$$(x + 4)^2 + 1$$

is always positive, no matter what the value of x.

Question 8.

By completing the square, show that the expression

$$x^2 + 2x + 5$$

is positive for all real values of *x*.

Question 9.

$$f(x) = x^2 + (k + 5)x + 4k$$

where *k* is a real constant.

(i) Show that the discriminant can be expessed in the form

$$(k - a)^2 + b$$

where *a* and *b* are positive integers.

(ii) Explain how your part **(i)** answer reveals that $f(x) = 0$ will always have two real roots

Question 10.
How many roots does the following equation have ?

$$6x^2 + 7x - 5 = 0$$

Justify your answer.

Question 11.
Consider the equation

$$(k + 1)x^2 + kx + k + 1 = 0$$

where k is a constant.

This equation has a repeated root.

Determine the possible values of k.

6.3 Homework

Question 1.

Factorise completely;

(i) $x^2 + 11x + 28$

(ii) $x^2 + x - 30$

(iii) $3x^2 - 18x + 24$

(iv) $x^2 + 13x + 36$

Question 2.

Solve the following equations;

(i) $x^2 + 7x - 8 = 0$

(ii) $3x^2 + 17x + 10 = 0$

(iii) $x^2 - 2x - 35 = 0$

(iv) $4x^2 - 24x + 32 = 0$

Question 3.

Find the value of,

(i) $25^{\frac{1}{2}}$

(ii) $27^{\frac{2}{3}}$

(iii) 2^{-3}

Question 4.
Write each of the following in the form $a\sqrt{b}$ where a and b are integers and b is also square free;

(i) $\sqrt{45}$ **(ii)** $\dfrac{6}{\sqrt{2}}$ **(iii)** $\dfrac{\sqrt{108}}{\sqrt{6}}$

Question 5.
Rationalise the denominator of;

$$\frac{2 + \sqrt{7}}{1 - \sqrt{7}}$$

Question 6.
(i) Find the values of a, b and c for which $5x^2 - 35x + 6 \equiv a(x + b)^2 + c$

(ii) Hence, state the minimum value of the function $f(x) = 5x^2 - 35x + 6$.

Question 7.
(i) Find the values of a, b and c for which $4x^2 - 12x + 25 \equiv a(x + b)^2 + c$

(ii) Hence, state the minimum value of the function $f(x) = 4x^2 - 12x + 25$.

6.4 Answers

6.4.1 Solutions to 6.2 Exercise

Answer 1.

(i)	D = 28 which is +ve	$\therefore$ 2 distinct real roots
(ii)	D = - 28 which is -ve	$\therefore$ 0 real roots
(iii)	D = 0	$\therefore$ 1 real root
(iv)	D = - 3 which is -ve	$\therefore$ 0 real roots
(v)	D = - 11 which is -ve	$\therefore$ 0 real roots
(vi)	D = 0	$\therefore$ 1 real root

Answer 2.

No real roots $\Rightarrow$

$$\begin{aligned} D &< 0 \\ (-3)^2 - 4 \times 1 \times (-k) &< 0 \\ 9 + 4k &< 0 \\ 4k &< -9 \\ k &< -\frac{9}{4} \\ k &< -2.25 \end{aligned}$$

Answer 3.

(i) Two distinct real roots $\Rightarrow$

$$\begin{aligned} D &> 0 \\ k^2 - 4 \times 1 \times 4 &> 0 \\ k^2 - 16 &> 0 \\ k^2 &> 16 \end{aligned}$$

(ii)

$$\begin{aligned} x^2 + 5x + 4 &= 0 \quad && \textit{Yes, this has 2 roots} \\ x^2 + 3x + 4 &= 0 && \textit{No, this does not have 2 roots} \\ x^2 - x + 4 &= 0 && \textit{No, this does not have 2 roots} \\ x^2 - 7x + 4 &= 0 && \textit{Yes, this has 2 roots} \end{aligned}$$

(iii) $-3 \pm \sqrt{5}$

Answer 4.

For a repeated root ⇒

$$\begin{aligned} D &= 0 \\ (2k+10)^2 - 4 \times 1 \times (k^2+5) &= 0 \\ 4k^2 + 40k + 100 - 4k^2 - 20 &= 0 \\ 40k + 80 &= 0 \\ k &= -2 \end{aligned}$$

Answer 5.

For a repeated root ⇒

$$\begin{aligned} D &= 0 \\ 144 - 4k^2 &= 0 \\ k^2 &= 36 \\ k &= \pm 6 \\ k &= 6 \quad \textit{as question states k is positive} \end{aligned}$$

Answer 6.

(a) 4 **(b)** - 4

Answer 7.

$\textit{Obviously}, \ (x+4)^2 \geqslant 0$

$\therefore \ (x+4)^2 + 1 > 0 \qquad \textit{i.e. Is always } + ve$

Answer 8.

$$\begin{aligned} & x^2 + 2x + 5 \\ &= (x+1)^2 - 1 + 5 \\ &= (x+1)^2 + 4 \end{aligned}$$

$$\textit{As } (x+1)^2 \textit{ is always } + ve \textit{ or equal to zero}$$

$$\Rightarrow (x+1)^2 + 4 \textit{ is always positive}$$

$$\Rightarrow x^2 + 2x + 5 \quad \textit{is also always } + ve$$

Answer 9.

(i)

$$\begin{aligned} D &= (k+5)^2 - 4 \times 1 \times 4k \\ &= k^2 + 10k + 25 - 16k \\ &= k^2 - 6k + 25 \\ &= (k-3)^2 - 9 + 25 \\ &= (k-3)^2 + 16 \end{aligned}$$

(ii)

$\textit{Obviously}, \quad (k - 3)^2 \geqslant 0$

$\therefore \quad (x - 3)^2 + 16 \quad \geqslant 0$

$\therefore \quad D \textit{ is always } + ve$

$\therefore \quad f(x) = x^2 + (k + 5)x + 4k \textit{ will always have two real roots}$

Answer 10.

D = 169 which is +ve, so 2 real roots

Answer 11.

A repeated root ⇒

$$\begin{aligned} D &= 0 \\ k^2 - 4 \times (k + 1)(k + 1) &= 0 \\ k^2 - 4(k^2 + 2k + 1) &= 0 \\ k^2 - 4k^2 - 8k - 4 &= 0 \\ -3k^2 - 8k - 4 &= 0 \\ 3k^2 + 8k + 4 &= 0 \\ (3k + 2)(k + 2) &= 0 \end{aligned}$$

$$\therefore k = -\frac{2}{3} \quad or \quad k = -2$$

Chapter 7

Algebra : Core 1

7.1 Sketching Quadratic Curves.

Given a quadratic curve, our various algebraic techniques are useful in quickly getting an approximate idea of what its graph looks like, without going to the time and effort of plotting an accurate graph.

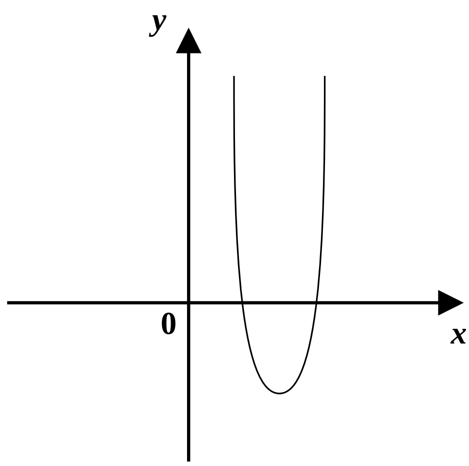
7.2 Example

Consider the curve:

$$y = x^2 - 14x + 46$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



(iii) What is the minimum value of the function $f(x) = x^2 - 14x + 46$?

(iv) For what value of x does the function $f(x) = x^2 - 14x + 46$ have minimum value ?

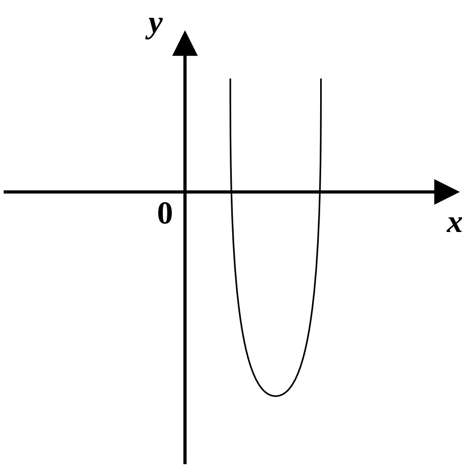
7.3 Exercise.

Question 1.
Consider the curve:

$$y = x^2 - 8x + 5$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



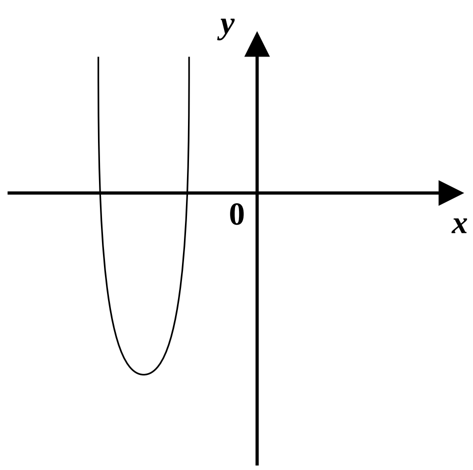
(iii) What is the minimum value of the function $f(x) = x^2 - 8x + 5$?

Question 2.
Consider the curve:

$$y = x^2 + 10x + 18$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



(iii) For what value of x does the function $f(x) = x^2 + 10x + 18$ have minimum value ?

Question 3.
Without sketching a graph, for each function state it's minimum value.

(i)

$$f(x) = (x - 3)^2 + 11$$

(ii)

$$f(z) = \left(z + \frac{1}{4}\right)^2 + \frac{3}{8}$$

(iii)

$$g(x) = 8(x + 7)^2 - 1$$

(iv)

$$f(w) = 4\left(w - \frac{5}{2}\right)^2 - \frac{3}{4}$$

Question 4.
Without sketching a graph, for each function state the value of *x* for which the function has minimum value.

(i)

$$f(x) = (x - 8)^2 + 13$$

(ii)

$$f(x) = \left(x - \frac{2}{5}\right)^2 + \frac{4}{7}$$

(iii)

$$g(x) = 7(x + 2.4)^2 - 3.6$$

(iv)

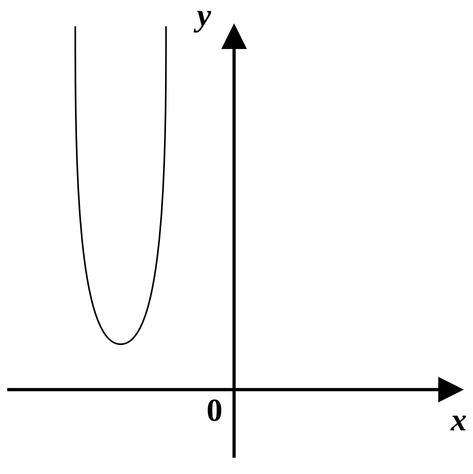
$$f(x) = \frac{3}{10}\pi(x - 8)^2 - 32$$

Question 5.
Consider the curve:

$$y = 3x^2 + 30x + 76$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



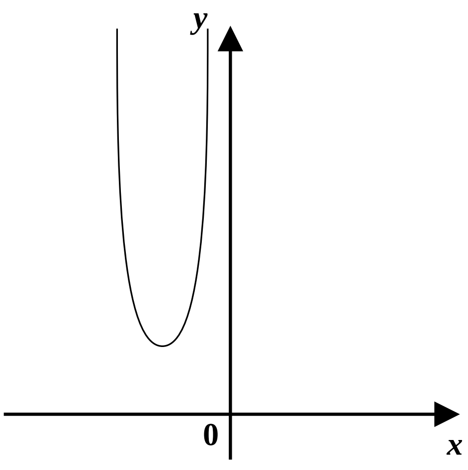
(iii) What is the minimum value of the function $f(x) = 3x^2 + 30x + 76$?

Question 6.
Consider the curve:

$$y = x^2 + x + 1$$

(i) Rewrite the equation of the curve in completed square form.
Your answer will have some fractions in it.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



(iii) For what value of x does the function $f(x) = x^2 + x + 1$ have minimum value ?

Question 7.

Consider the curve;

$$y = x^2 - 5x - 24$$

In this question we are NOT interested in finding the minimum point.
Instead we wish to know where the graph of this curve crosses the *x*-axis.

(i) Write the equation of the curve in factorised form.

(ii) Hence state the coordinates of the two points where the graph of the curve crosses the *x*-axis.

(iii) Without further calculation, sketch the curve.

7.4 Homework

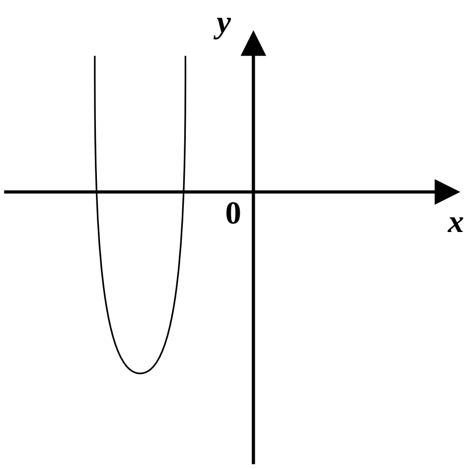
Question 1.

Consider the curve:

$$y = x^2 + 10x + 13$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



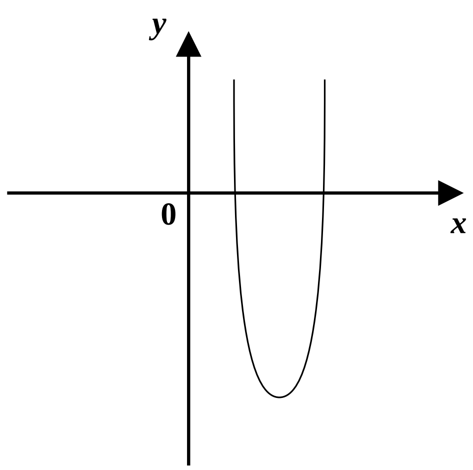
(iii) What is the minimum value of the function $f(x) = x^2 + 10x + 13$?

Question 2.
Consider the curve:

$$y = x^2 - 12x - 21$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence add the coordinates of the minimum point to the sketch graph.



(iii) For what value of x does the function $f(x) = x^2 - 12x - 21$ have minimum value ?

Question 3.
Consider the curve:

$$y = 4x^2 + 24x + 11$$

(i) Rewrite the equation of the curve in completed square form.

(ii) Hence sketch the graph of the curve with the coordinates of its minimum point clearly marked

7.5 Answers

7.5.1 Solution to Introductory Example (7.2 Example)

(i)

$$y = (x - 7)^2 - 3$$

(ii) $(7, -3)$

(iii) -3

(iv) 7

7.5.2 Solutions to Exercise (7.3 Exercise)

Answer 1.

(i)

$$y = (x - 4)^2 - 11$$

(ii) $(4, -11)$

(iii) -11

Answer 2.

(i)

$$y = (x + 5)^2 - 7$$

(ii) $(-5, -7)$

(iii) -5

Answer 3.

(i) 11

(ii) $\frac{3}{8}$

(iii) -1

(iv) $-\frac{3}{4}$

Answer 4.

(i) 8

(ii) $\frac{2}{5}$

(iii) -2.4

(iv) 8

Answer 5.

(i)

$$y = 3(x + 5)^2 + 1$$

(ii) $(-5, 1)$

(iii) 1

Answer 6.

(i)

$$y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

(ii)

$$\left(-\frac{1}{2}, \frac{3}{4} \right)$$

(iii)

$$-\frac{1}{2}$$

Answer 7.

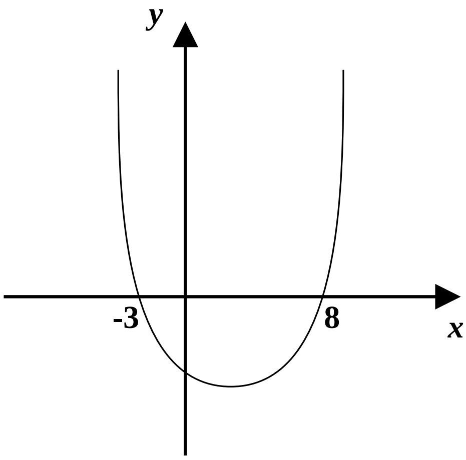
(i)

$$y = (x + 3)(x - 8)$$

(ii)

$(-3,\ 0)$ & $(8,\ 0)$

(iii)



Chapter 8

8.1 Inequalities.

Examples

Solve these inequalities:

(i) $5x + 2 \leqslant 9$

(ii) $7 - 3x \geqslant 13$

(iii) $-2 < 4x + 3 \leqslant 9$

(iv) $\frac{4}{x} < \frac{5}{9}$

(v) $x^2 - 2x - 15 < 0$

8.2 Exercise

Question 1.

Solve these inequalities;

(i) $7x - 3 > 25$

(ii) $13 - 4x < 11$

(iii) $3x - 5 > 19 - 5x$

(iv) $3(2x + 3) - 4(x - 2) < 11$

(v) $17 + 7(x - 6) > 45$

(vi) $13 < 5x + 3 < 21$

(vii) $-3 < 5 - 4x \leqslant 25$

(viii) $\frac{10}{x} < \frac{5}{12}$

Question 2.

Solve these quadratic inequalities by;

- ☐ First factorising the quadratic into two pairs of brackets.
- ☐ Clearly stating the 'critical values' of the quadratic.
- ☐ Sketching the quadratic.
- ☐ Presenting your final solution with inequality signs clearly displayed.

(i) $x^2 + 9x + 20 \leqslant 0$

(ii) $x^2 + 3x - 18 < 0$

(iii) $x^2 + 13x + 12 > 0$ **(iv)** $x^2 - 11x + 24 \geqslant 0$

Question 3.

The discriminant of the quadratic equation $a x^2 + bx + c = 0$ is $D = b^2 - 4ac$.
Each of the following eqauations is to have two distinct real roots.
$\therefore$ $D > 0$.
By considering this inequality, find the range of k.

(i) $x^2 + 6x + k = 0$

(ii) $k x^2 + 4x + 2 = 0$

(iii) $x^2 - 2x + k = 0$ **(iv)** $kx^2 - 6x + 3 = 0$

Question 4.

Consider the following equation;

$$k\,x^2 - 2x + k = 0$$

This equation has two distinct real roots.

(i) By considering the discriminant show that this implies that,

$$(k - 1)(k + 1) < 0$$

(ii) Hence state the range of values of *k*.

Question 5

(i) By completing the square, write the algebraic expression

$$2x^2 - 28x + 87$$

in the form

$$a(x + b)^2 + c$$

where a, b and c are numbers to be found.

(ii) Hence show that the equation

$$2x^2 - 28x + 87 = 0$$

has solutions of the form

$$x = c \pm d\sqrt{22}$$

where c and d are numbers to be found.

Chapter 9

Algebra : Core 1

9.1 Simultaneous Equations (One linear, one Quadratic)

Questions that ask you to solve simultaneous equations where one equation is linear and the other is quadratic often come up in the C1 examination because they test several algebraic manipulation skills in one go...

- Expanding brackets, FOIL
- Gathering together like terms
- Rearranging equations into the form $f(x) = 0$
- Factorising quadratics
- Solving quadratic equations

Example : The Question

Solve the simultaneous equations

$$y = 2x - 3 \qquad (\textit{line})$$

$$x^2 + y^2 = 2 \qquad (\textit{circle})$$

Example : The Solution

We begin by using *the method of substitution.*

$$x^2 + y^2 = 2$$

$$x^2 + (2x - 3)^2 = 2$$

- Expanding brackets, FOIL

$$x^2 + (2x - 3)(2x - 3) = 2$$

$$x^2 + 4x^2 - 6x - 6x + 9 = 2$$

- Gathering together like terms

$$5x^2 - 12x + 9 = 2$$

- Rearranging equations into the form $f(x) = 0$

$$5x^2 - 12x + 7 = 0$$

- Factorising quadratics

$$(5x - 7)(x - 1) = 0$$

- Solving quadratic equations

$$EITHER \ 5x - 7 = 0 \quad OR \quad x - 1 = 0$$

$$x = 1.4 \qquad\qquad x = 1$$

The answer is points where the line intersects the circle so we finally use the equation of the line $y = 2x - 3$ to work out y when x is 1.4, and 1.

FINAL ANSWER:

$$(1.4, \ -0.2) \ \textit{and} \ (1, \ -1)$$

9.2 Exercise

Question 1.

Solve the simultaneous equations

$$y = x - 4$$
$$x^2 + y^2 = 58$$

Question 2.
GCSE, June 2007, paper 3H, Q19
Solve the simultaneous equations

$$y = 3x - 1$$
$$x^2 + y^2 = 5$$

Question 3.

C1 Examination question from May 2011, Q4.

Solve the simultaneous equations

$$x + y = 2$$

$$4y^2 - x^2 = 11$$

Question 4.

Solve the simultaneous equations

$$y = x - 7$$

$$x^2 + y^2 = 109$$

Question 5.

C1 Examination question from January 2010, Q5.

$$y - 3x + 2 = 0$$

$$y^2 - x - 6x^2 = 0$$

Question 6.
Solve the simultaneous equations

$$y = 2x - 3$$
$$x^2 + y^2 = 18$$

Chapter 10

Algebra : Core 1

10.1 Past Paper Work

The algebraic "standard, routine techniques" that we've been looking at feature prominantly in the C1 examination. The next exercise is constructed entirely from old examination questions and lets you see how the material we've covered is tested in the exam.

10.2 Exercise

Question 1.

C1 Examination question from January 2012, Q3.

Find the set of values of x for which

(a)

$4x - 5 > 15 - x$

[2 marks]

(b)

$x(x - 4) > 12$

[4 marks]

Question 2.

C1 Examination question from June 2008, Q2.

Factorise completely

$$x^3 - 9x$$

[3 marks]

Question 3.

C1 Examination question from June 2009, Q4.

Find the set of values of x for which

(a)

$4x - 3 > 7 - x$

[2 marks]

(b)

$2x^2 - 5x - 12 < 0$

[4 marks]

(c)

both $4x - 3 > 7 - x$ ***and*** $2x^2 - 5x - 12 < 0$

[1 mark]

Question 4.

C1 Examination question from May 2010, Q3.

Find the set of values of x for which

(a)

$3(x - 2) < 8 - 2x$

[2 marks]

(b)

$(2x - 7)(1 + x) < 0$

[3 marks]

(c)

both $3(x - 2) < 8 - 2x$ ***and*** $(2x - 7)(1 + x) < 0$

[1 mark]

Question 5.

C1 Examination question from May 2010, Q4.

(a) Show that $x^2 + 6x + 11$ can be written as

$$(x + p)^2 + q$$

where p and q are integers to be found.

[2 marks]

(b) Sketch the curve with equation $y = x^2 + 6x + 11$ showing clearly any intersections with the coordinate axes.

[2 marks]

(c) Find the value of the discriminant of $x^2 + 6x + 11$

[2 marks]

Question 6.

C1 Examination question from June 2008, Q8.

Given that the equation $2qx^2 + qx - 1 = 0$, where q is a constant, has no real roots,

(a) Show that

$$q^2 + 8q < 0$$

[2 marks]

(b) Hence find the set of possible values of q.

[3 marks]

Question 7.

C1 Examination question from May 2011, Q7.

$$f(x) = x^2 + (k + 3)x + k$$

where k is a real constant.

(a) Find the discriminant of $f(x)$ in terms of k.

[2 marks]

(b) Show that the discriminant of $f(x)$ can be expressed in the form $(k + a)^2 + b$ where a and b are integers to be found.

[2 marks]

(c) Show that, for all values of k, the equation $f(x) = 0$ has real roots.

[2 marks]

Question 8.

C1 Examination question from January 2009, Q7.

The equation $k x^2 + 4x + (5 - k) = 0$, where k is a constant, has 2 different real solutions for x.

(a) Show that k satisfies

$$k^2 - 5k + 4 > 0$$

[3 marks]

(b) Hence find the set of possible values of k.

[4 marks]

Question 9.

C1 Examination question from January 2007, Q4.

Solve the simultaneous equations

$$y = x - 2$$

$$y^2 + x^2 = 10$$

[7 marks]

Question 10.

C1 Examination question from January 2008, Q8.

The equation

$$x^2 + kx + 8 = k$$

has no real solutions for x.

(a) Show that k satisfies $k^2 + 4k - 32 < 0$

[3 marks]

(b) Hence find the set of possible values of k.

[4 marks]

Question 11.

C1 Examination question from January 2007, Q5.

The equation $2x^2 - 3x - (k + 1) = 0$, where k is a constant, has no real roots.

Find the set of possible values of k.

[4 marks]

Question 12.

C1 Examination question from January 2010, Q10.

$$f(x) = x^2 + 4kx + (3 + 11k)$$

where k is a constant.

(a) Express $f(x)$ in the form $(x + p)^2 + q$, where p and q are constants to be found in terms of k.

[3 marks]

Given that the equation $f(x) = 0$ has no real roots,

(b) Find the set of possible values of k.

[4 marks]

Given that $k = 1$,

(c) Sketch the graph of $y = f(x)$, showing the coordinates of any point at which the graph crosses a coordinate axis.

[3 marks]

Question 13.

C1 Examination question from January 2007, Q10.

(a) On the same axes sketch the graphs of the curves with equations

(i) $y = x^2 (x - 2)$

[3 marks]

(ii) $y = x (6 - x)$

[3 marks]

and indicate on your sketches the coordinates of all the points where the curves cross the x-axis.

(b) Use algebra to find the coordinates of the points where the graphs intersect.

[7 marks]

Chapter 11

Algebra : Core 1

11.1 More Past Paper Work

Question 1.

C1 Examination question from January 2011, Q8.

The equation

$$x^2 + (k - 3)x + (3 - 2k) = 0$$

where k is a constant, has two distinct real roots.

(a) Show that k satisfies

$$k^2 + 2k - 3 > 0$$

[3 marks]

(b) Find the set of possible values of k.

[4 marks]

Question 2.

C1 Examination question from May 2007, Q6.

(a) By eliminating y from the equations

$$y = x - 4$$

$$2x^2 - xy = 8$$

show that

$$x^2 + 4x - 8 = 0$$

[2 marks]

(b) Hence, or otherwise, solve the simultaneous equations

$$y = x - 4$$

$$2x^2 - xy = 8$$

giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers.

[5 marks]

Question 3.

C1 Examination question from May 2007, Q7.

The equation

$$x^2 + kx + (k + 3) = 0$$

where k is a constant, has different real roots.

(a) Show that

$$k^2 - 4k - 12 > 0$$

[2 marks]

(b) Find the set of possible values of k.

[4 marks]

Question 4.

The width of a rectangular sports pitch is x metres, $x > 0$.
The length of the pitch is 20 metres more than its width.
The perimeter of the pitch must be less than 300 metres.

(a) Form a linear inequality in x.

[2 marks]

Given that the area of the pitch must be greater than 4800 m^2;

(b) Form a quadratic inequality in x.

[2 marks]

(c) By solving your inequalities, find the set of possible values of x.

[4 marks]

11.2 Homework

Question 1.

C1 Examination question from May 2014, Q3.

Find the set of values of x for which

(a) $3x - 7 > 3 - x$

[2 marks]

(b) $x^2 - 9x \leqslant 36$

[4 marks]

(c) **both** $3x - 7 > 3 - x$ **and** $x^2 - 9x \leqslant 36$

[1 mark]

Question 2.

C1 Examination question from May 2002, Q4.

(a) By completing the square, find in terms of k the roots of the equation

$$x^2 + 2kx - 7 = 0$$

[4 marks]

(b) Prove that, for all real values of k, the roots of $x^2 + 2kx - 7 = 0$ are real and different.

[2 marks]

(c) Given that $k = \sqrt{2}$ find the exact roots of the equation.

[2 marks]

Question 3.

C1 Examination question from January 2001, Q2.

(a) Prove, by completing the square, that the roots of the equation $x^2 + 2kx + c = 0$, where k and c are constants, are $-k \pm \sqrt{k^2 - c}$.

[4 marks]

The equation $x^2 + 2kx + 81 = 0$ has equal roots.

(b) Find the possible values of k.

[2 marks]

Chapter 12

Algebra : Core 1

12.1 Past Paper Questions Involving Simultaneous Equations

Question 1.

C1 Examination question from January 2005, Q4.

Solve the simultaneous equations

$$x + y = 2$$

$$x^2 + 2y = 12$$

[6 marks]

Question 2.

C1 Examination question from May 2005, Q5.

Solve the simultaneous equations

$$x - 2y = 1$$

$$x^2 + y^2 = 29$$

[6 marks]

Question 3.

C1 Examination question from June 2008, Q6.

The curve C has equation $y = \frac{3}{x}$ and the line l has equation $y = 2x + 5$

(a) On one set of axes, sketch the graphs of both C and l, indicating clearly the coordinates of any intersections with the axes.

[3 marks]

(b) Find the coordinates of the points of intersection of C and l

[6 marks]

Question 4.

C1 Examination question from January 2007, Q10.

(a) On the same axes sketch the graphs of the curves with equations

(i) $y = x^2 (x - 2)$

[3 marks]

(ii) $y = x (6 - x)$

[3 marks]

and indicate on your sketches the coordinates of all points where the curves cross the *x*-axis.

(b) Use algebra to find the coordinates of the points where the graphs intersect.

[7 marks]

Question 5.

C1 Examination question from May 2010, Q10.

(a) On the same axes sketch the graphs of the curves with equations

(i) $y = x(4 - x)$

(ii) $y = x^2(7 - x)$

showing clearly the coordinates of the points where the curves cross the coordinate axes.

[5 marks]

(b) Show that the x-coordinate of the points of intersection of

$$y = x(4 - x) \quad \textit{and} \quad y = x^2(7 - x)$$

are given by the solutions to the equation

$$x(x^2 - 8x + 4) = 0$$

[3 marks]

Continues over....

... continued ...

The point *A* lies on both the curve and the *x* and *y* coordinate of *A* are both positive.

(c) Find the exact coordinates of *A*, leaving your answer in the form

$$(p + q\sqrt{3},\ r + s\sqrt{3})$$

where *p*, *q*, *r* and *s* are integers.

[7 marks]

Chapter 13

Algebra : Core 1

13.1 Graphing The Algebra

Review of Graph Questions from Exercise 12.1, Questions 3 (a), 4 (a) & 5 (a)

13.2 Together Sketches

Sketch the shape the graph for each of the following curves, marking on all axis crossing points.

$y = x^2$

$y = (x + 3)(x - 1)^2$

$y = x^3$

$y = (x + 3)(4 - x)$

13.3 Exercise

Sketch the shape the graph for each of the following curves, marking on all axis crossing points.

$y = (x + 4)(x - 1)$

$y = x^2 + 9x + 20$

$y = (x + 5)^2$

$y = (x + 4)(x + 1)(x - 3)$

$$y = (x+1)^2 (x-5)$$

$$y = \frac{1}{x}$$

Hint: Inverse proportion, 'more is less'

$$y = \frac{1}{x^2}$$

$$y = x^3 - x$$

$y = (x - 4)^2 - 9$

Hint: Extract minimum point

Exand brackets

Factorise into (....) (....)

$y = (x - 3)^2 + 1$

Hint : Extract minimum point

How many *x*-axis crossings ?

Expand to get *y*-axis crossing

$y = -x^2$

$y = (x + 3)(5 - x)$

$y = (x + 3)^2 (2 - x)$

$y = -\frac{1}{x}$

$y = \frac{1}{x^2}$

$y = x - x^3$

14.1 More Graphing The Algebra

14.2 Exercise

Question 1.

Sketch the shape the graph for each of the following curves, marking on all axis crossing points.

$y = (x + 2)^2 (x - 3)$

$y = \dfrac{12}{x}$

Hint: Inverse proportion, 'more is less'
12 does not change the shape

$y = (x + 2)(6 - x)$

$y = (x + 4)(1 - x)^2$

Hint : Careful. This'll be the $+x^3$ shape

$y = x(x-3)^2$

$y = -\frac{8}{x}$

$y = (x-1)^2 - 16$

Hint: Extract minimum point

Exand brackets

Factorise into (....) (....)

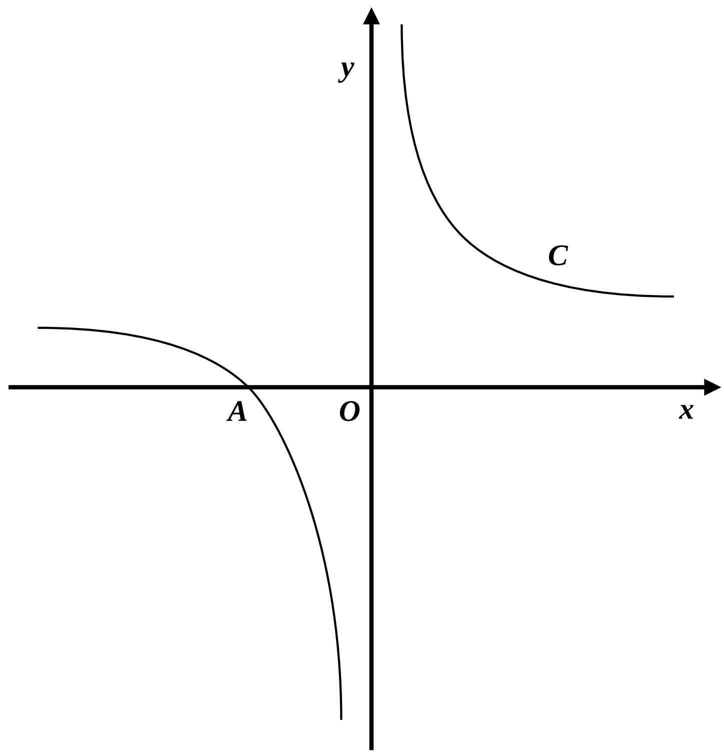
$y = (x-4)^2 + 1$

Hint : Extract minimum point

How many *x*-axis crossings ?

Expand to get *y*-axis crossing

Queation 2.
C1 examination question from May 2014, Q4



The diagram shows a sketch of the curve C with equation

$$y = \frac{1}{x} + 1 \qquad x \neq 0$$

The curve C crosses the x-axis at the point A.

(a) State the x coordinate of the point A.

[1 mark]

The curve D has equation

$$y = x^2 (x - 2)$$

for all real values of x.

(b) Add a sketch of curve D to the diagram, above.
Show the coordinates of each point where the curve D crosses the axes.

[3 marks]

(c) Using your sketch, state, giving a reason, the number of real solutions to the equation

$$x^2 (x - 2) = \frac{1}{x} + 1$$

[1 mark]

Queation 3.

C1 examination question from January 2009, Q8

The point P (1, a) lies on the curve wuth equation

$$y = (x + 1)^2 (2 - x)$$

(a) Find the value of a

[1 mark]

(b) On the axes below sketch the curves with the following equations;

(i) $y = (x + 1)^2 (2 - x)$

(ii) $y = \frac{2}{x}$

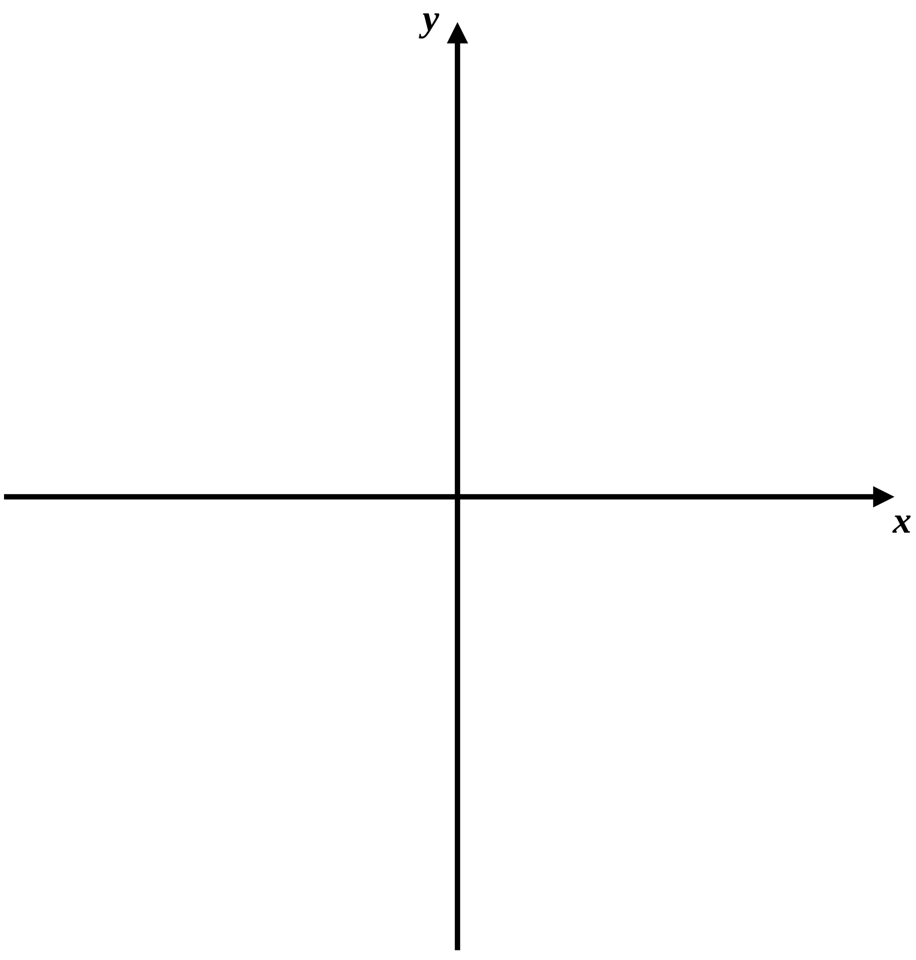
On your diagram show clearly the cordinates of any points at which the curves meet the axes.

[5 marks]

(c) With reference to your diagram in part (b) state the number of real solutions to the equation

$$(x + 1)^2 (2 - x) = \frac{2}{x}$$

[1 mark]



Queation 4.

C1 examination question from January 2011, Q10

(a) Sketch the graphs of

(i) $y = x(x+2)(3-x)$

(ii) $y = -\dfrac{2}{x}$

Show clearly the coordinates of all points where the curves cross the coordinate axes.

[6 marks]

(b) Using your sketch state, giving a reason, the number of real solutions to the equation

$$x(x+2)(3-x) = -\frac{2}{x}$$

[2 marks]

Queation 5.

C1 examination question from January 2006, Q10

$$x^2 + 2x + 3 \equiv (x + a)^2 + b$$

(a) Sketch Find the values of the constants a and b

[2 marks]

(b) In the space provided below, sketch the graph of

$$y = x^2 + 2x + 3$$

Indicate clearly the coordinates of any intersections with the coordinate axes.

[3 marks]

(c) Find the value of the discriminant of $x^2 + 2x + 3$
Explain how the sign of the discriminant relates to your sketch in part (b)

[2 marks]

The equation $x^2 + kx + 3 = 0$, where k is a constant, has no real roots.

(d) Find the set of possible values of k, giving your answer in surd form.

[4 marks]

Queation 6.

C1 examination question from January 2005, Q10

Given that

$$f(x) = x^2 - 6x + 18, \qquad x \geqslant 0$$

(a) express $f(x)$ in the form $(x - a)^2 + b$, where a and b are integers.

[3 marks]

The curve C with equation $y = f(x)$, $x \geqslant 0$, meets the y-axis at P and has a minimum point at Q

(b) Sketch the graph of C, showing the coordinates of P and Q

[4 marks]

The line $y = 41$ meets C at the point R

(c) Find the coordinates of R, giving your answer in the form $p + q\sqrt{2}$ where p and q are integers.

[5 marks]