Number Theory I

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

Number Theory I

Number Theory : Year 9

Chapter 1

1.1 Primes

A prime number is a positive integer with two factors exactly.
A positive integer with more than two factors is said to be composite.
The number 1 is neither prime nor composite.
There are twenty-five primes less than 100.

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

Primes are important because the composite numbers are built from primes.
Primes are the atoms of the mathematical universe.

1.2 The fundamental theorem of arithmetic

All composite numbers can be written as a product of primes.
The representation of any composite number in this way is essentially unique.

1.2.1 Examples

(i) $50 = 2 \times 5 \times 5$

(ii) $49 = 7 \times 7$

(iii) $19669 = 13 \times 17 \times 89$

1.3 Exercise.

Write each of the following composite numbers as a product of prime numbers.

(i) 8 =

(ii) 10 =

(iii) 18 =

(iv) 50 =

(v) 30 =

(vi) 35 =

(vii) 70 =

(viii) 6 =

(ix) 100 =

(x) 121 =

(xi) 66 =

(xii) 52 =

(xiii) 98 =

1.4 Writing larger composites as products of primes.

There are two popular ways of doing this..
The cherry tree method or the division ladder method.

1.4.1 Example - the devil's number : 666

1.5 Exercise.

Question 1.
Use either the cherry tree method or the division ladder method to write each of the following composite numbers as a product of primes.

HINT : Keep an eye of the twenty-five primes less than 100.

(i) 564 **(ii)** 1550 **(iii)** 936

Question 2.
Use either the cherry tree method or the division ladder method to write each of the following composite numbers as a product of primes.

HINT : Keep an eye of the twenty-five primes less than 100.

(i) 1375 **(ii)** 6264

Question 3.
In this question you can use a calculator.
Use either the cherry tree method or the division ladder method to write each of the following composite numbers as a product of primes.

(i) 213928 **(ii)** 1043504

Question 4.

In this question you can use a calculator.

Use the number grid below to help work out which numbers between 101 and 200 are prime numbers.

HINT : There are 21 to be found.

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

Question 5 : The Goldbach Conjecture.

Every even number, other than 2, can be written as the sum of two primes.

This may or may not be true - it's a famous unresolved statement.

Examples : 4 = 2 + 2, 6 = 3 + 3, and 8 = 3 + 5.

Write each of the following even numbers as the sum of two primes

(i) 8 =

(ii) 20 =

(iii) 36 =

(iv) 70 =

(v) 98 =

(vi) 128 =

(vii) 162 =

(viii) 174 =

(ix) 196 =

(x) 200 =

Question 6 : Fermat's triangular triples.

In 1636, Fermat claimed that

Every integer is the sum of three triangular numbers.

This was later proved to be true by Gauss.

In the following examples, note that 0 is considered to be a triangular number .

1 = 0 + 0 + 1						
2 = 0 + 1 + 1						
3 = 1 + 1 + 1	or	0 + 0 + 3				
4 =		0 + 1 + 3				
5 =		1 + 1 + 3				
6 =			0 + 3 + 3	or	0 + 0 + 6	
7 =			1 + 3 + 3	or	0 + 1 + 6	
8 =					1 + 1 + 6	

Continue the above examples by writing each the numbers from 9 to 20 as the sum of three triangular numbers.

9 =

10 =

11 =

12 =

13 =

14 =

15 =

16 =

17 =

18 =

19 =

20 =

Question 7 : Sum of two squares primes.

Another of Fermat's famous theorems says that :

If a prime has remainder 1 when divided by 4,
then that prime can be written as a sum of two squares.

Example

Consider the prime 41.
When divided by four this has remainder 1.
Thus 41 can be written as the sum of two squares.
After a bit of thought... $41 = 5^2 + 4^2$

Each of the following primes has remainder 1 when divided by 4.
Therefore they can be written as the sum of two squares.
In each case, find the two squares.

5 =

13 =

17 =

29 =

37 =

53 =

61 =

73 =

89 =

97 =

1.6 Answers

1.6.1 Solutions (1.3 Exercise)

(i)	8	$2 \times 2 \times 2$	2^3
(ii)	10	2×5	
(iii)	18	$2 \times 3 \times 3$	2×3^2
(iv)	50	$2 \times 5 \times 5$	2×5^2
(v)	30	$2 \times 3 \times 5$	
(vi)	35	5×7	
(vii)	70	$2 \times 5 \times 7$	
(viii)	6	2×3	
(ix)	100	$2 \times 2 \times 5 \times 5$	$2^2 \times 5^2$
(x)	121	11×11	11^2
(xi)	66	$2 \times 3 \times 11$	
(xii)	52	$2 \times 2 \times 13$	$2^2 \times 13$
(xiii)	98	$2 \times 7 \times 7$	2×7^2

1.6.2 Solutions (1.4 Example)

	666	$2 \times 3 \times 3 \times 37$	$2 \times 3^2 \times 37$

1.6.3 Solutions (1.5 Exercise)

Answer 1.

(i)	564	$2 \times 2 \times 3 \times 47$	$2^2 \times 3 \times 47$
(ii)	1550	$2 \times 5 \times 5 \times 31$	$2 \times 5^2 \times 31$
(iii)	936	$2 \times 2 \times 2 \times 3 \times 3 \times 13$	$2^3 \times 3^2 \times 13$

Answer 2.

(i)	1375	$5 \times 5 \times 5 \times 11$	$5^3 \times 11$
(ii)	6264	$2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 29$	$2^3 \times 3^3 \times 29$

Answer 3.

(i)	213928	$2 \times 2 \times 2 \times 11 \times 11 \times 13 \times 17$	$2^3 \times 11^2 \times 13 \times 17$
(ii)	1043504	$2 \times 2 \times 2 \times 2 \times 7 \times 7 \times 11 \times 11 \times 11$	$2^4 \times 7^2 \times 11^3$

An Amusement

Some calculators will incorrectly tell you that 2022196291 divides by 2

Chapter 2

2.1 *hcf* and *lcm*

2.1.1 Example : *highest common factor*.

The *highest common factor* of 6 and 15 is the biggest integer that divides into both 6 and 15 without remainder.

After a pause for thought : *hcf* {6,15} = 3.

2.1.2 Example : *lowest common multiple*.

The *lowest common multiple* of 6 and 15 is the first integer that is in the 6 times table and also in the 15 times table.

After a pause for thought : *lcm* {6,15} = 30.

2.1.3 The connection between *hcf* and *lcm*.

To work out an *lcm*, first find the *hcf*, then use;

$$lcm\{a,\ b\} \ = \ \frac{ab}{hcf\ \{a,\ b\}}$$

So, for example, to find *lcm* {18,33}.

First : Figure out that *hcf* {18,33} = 3.

Second : Work out $\frac{18 \times 33}{3}$ which is 198.

2.2 Exercise.

Question 1.

Write down the *hcf* and the *lcm* requested;

(i)	*hcf* {6, 10} =	*lcm* {6, 10} =
(ii)	*hcf* {2, 3}	*lcm* {2, 3} =
(iii)	*hcf* {12, 20} =	*lcm* {12, 20} =
(iv)	*hcf* {8, 16} =	*lcm* {8, 16} =
(v)	*hcf* {4, 9} =	*lcm* {4, 9} =
(vi)	*hcf* {26, 39} =	*lcm* {26, 39} =
(vii)	*hcf* {50, 80} =	*lcm* {50, 80} =
(viii)	*hcf* {30, 75} =	*lcm* {30, 75} =
(ix)	*hcf* {24, 36} =	*lcm* {24, 36} =
(x)	*hcf* {100, 105} =	*lcm* {100, 105} =
(xi)	*hcf* {8, 9} =	*lcm* {8, 9} =
(xii)	*hcf* {12, 16} =	*lcm* {12, 16} =
(xiii)	*hcf* {44, 66} =	*lcm* {44, 66} =

Question 2.

Given that $hcf\,\{245, 350\} = 35$, use the following formula to find $lcm\,\{245, 350\}$.

$$lcm\{a,\ b\} \;=\; \frac{ab}{hcf\,\{a,\ b\}}$$

Question 3.

I have a bag of sweets.
I could share it exactly between 15 people.
Or I could share it exactly between 27 people.

(i) What is $hcf\,\{15, 27\}$?

(ii) What is $lcm\,\{15, 27\}$?

(iii) What is the least number of sweets that must be in the bag ?

(iv) Will I definitely be able to share out the sweets exactly between 9 people ?

(v) Will I definitely be able to share out the sweets exactly between 45 people ?

Question 4.

DEFINITION
Two numbers are coprime when the only factor they have in common is 1.
i.e. If *hcf* {a, b} = 1 then *a* and *b* are coprime.

(i) Are 18 and 75 coprime ?

(ii) Are 32 and 45 coprime ?

(iii) Are 49 and 63 coprime ?

Question 5.
For each of the following pairs of integers, decide if they are coprime or not.

(i) 7 and 11

(ii) 15 and 21

(iii) 49 and 52

(iv) 10 and 11

(v) 13 and 39

(vi) 63 and 56

Question 6.

(a)
Write 4000 as a product of primes.

(b)
Write 3969 as a product of primes.

(c) Are 4000 and 3969 coprime ?

(d) Find *lcm*{4000, 3969} giving the answer as a product of primes.

Question 7.

Many mathematicians have searched for a formula that will generate some or all of the prime numbers. Unfortunately, although such formulae often initially seem to work well, a flaw seems to eventually always be found.

Here is one such formula.......

$$x^2 + x + 11 = prime$$

(i) Work out : $0^2 + 0 + 11 =$
Is the answer prime ?

(ii) Work out : $1^2 + 1 + 11 =$
Is the answer prime ?

(iii) Work out : $2^2 + 2 + 11 =$
Is the answer prime ?

(iv) Work out : $3^2 + 3 + 11 =$
Is the answer prime ?

(v) Work out : $4^2 + 4 + 11 =$
Is the answer prime ?

(vi) Work out : $5^2 + 5 + 11 =$
Is the answer prime ?

Now keep going. This formula is good enough to fool many people into thinking it will always give a prime number answer. Is it good enough to fool you ?

Question 8.

Here is a 'mad' number tower;

$$4^{2^3} = ?$$

To understand it, calculate the 2^3 part first.
$2^3 = 2 \times 2 \times 2 = 8$
Then $4^8 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 65536$.
So,

$$4^{2^3} = 4^8 = 65536$$

Make sure that you can get your calculator to give this answer.

In 1640, Fermat thought he had discovered a formula for generating primes. Here it is;

$$2^{2^x} + 1 = prime$$

(a) Find the numbers generated by this formula when;

(i) $x = 0$

(ii) $x = 1$

(iii) $x = 2$

(iv) $x = 3$

(v) $x = 4$

All the answers so far are prime.

(vi) $x = 5$

(b) This time the answer is not prime, although it would take a while to realise this as the smallest prime that divides into the answer is 641.

How many times does 641 divide into your part **(vi)** answer ?

Question 9.

Here is a "prime number making machine".
It attempts to use all known primes to generate a new, previously unknown prime.

Take the first *x* primes.
Multiply them all together.
Add 1.
The answer is a new prime.

(i) Work out $2 \times 3 + 1$
Is it prime ?

(ii) Work out $2 \times 3 \times 5 + 1$
Is it prime ?

(iii) Work out $2 \times 3 \times 5 \times 7 + 1$
Is it prime ?

(iv) Work out $2 \times 3 \times 5 \times 7 \times 11 + 1$
Is it prime ?

(v) Work out $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$
This answer is NOT prime.
What divides into it ?

2.4 Answers

2.4.1 Solutions (2.2 Exercise)

Question 1.

(i) 2 $\frac{6 \times 10}{2} = 3 \times 10 = 30$

(ii) 1 $\frac{2 \times 3}{1} = 2 \times 3 = 6$

(iii) 4 $\frac{12 \times 20}{4} = 3 \times 20 = 60$

(iv) 8 $\frac{8 \times 16}{8} = 1 \times 16 = 16$

(v) 1 $\frac{4 \times 9}{1} = 4 \times 9 = 36$

(vi) 13 $\frac{26 \times 39}{13} = 2 \times 39 = 78$

(vii) 10 $\frac{50 \times 80}{10} = 50 \times 8 = 400$

(viii) 15 $\frac{30 \times 75}{15} = 2 \times 75 = 150$

(ix) 12 $\frac{24 \times 36}{12} = 2 \times 36 = 72$

(x) 5 $\frac{100 \times 105}{5} = 100 \times 21 = 2100$

(xi) 1 $\frac{8 \times 9}{1} = 8 \times 9 = 72$

(xii) 4 $\frac{12 \times 16}{4} = 12 \times 4 = 48$

(xiii) 22 $\frac{44 \times 66}{22} = 2 \times 66 = 132$

Querstion 2.

$$\frac{245 \times 350}{35} = 245 \times 10 = 2450$$

Question 3.

(i) 3 **(ii)** $\frac{15 \times 27}{3} = 5 \times 27 = 135$

(ii) 135

(iv) Yes, at least 15 each

(v) Yes, at least 3 each

Question 4.

(i) No, *hcf* is 3, they both divide by 3.

(ii) Yes, *hcf* is 1, they have no common factor other than 1.

(iii) No, *hcf* is 7, they both divide by 7.

Question 5.

(i) Coprime

(ii) Not coprime, both divide by 3.

(iii) Coprime

(iv) Coprime

(v) Not coprime, both divide by 13

(vi) not coprime, both divide by 7.

Question 6.

(a)

$$4000 = 2^5 \times 5^3$$

(b)

$$3969 = 3^4 \times 7^2$$

(c) They are coprime as they have no factors in common

(d)

$$4000 \times 3969 = 2^5 \times 5^3 \times 3^4 \times 7^2$$

Chapter 3

Number Theory : Year 9

Calculator needed

3.1 Finding the *hcf* and *lcm* of pairs of large numbers

As the number pairs get bigger it becomes harder to mentally spot what the *hcf* and the *lcm* are. A system is needed to methodically work them out.

3.2 Strategy

To find the *hcf* and *lcm* of *a* and *b*.

- ◇ **STEP 1 :** Write each of *a* and *b* as a products of primes.
- ◇ **STEP 2 :** The *hcf* $\{a, b\}$ is all the **<u>common</u>** primes multiplied together.
- ◇ **STEP 3 :** The *lcm* $\{a, b\}$ is $\frac{a\,b}{hcf\{a,b\}}$

3.3 Example

Find the *hcf* and *lcm* of 210 and 245.

$$210 = 2 \times 3 \times 5 \times 7$$

$$245 = 5 \times 7 \times 7$$

$$hcf = 5 \times 7 \qquad \therefore\ hcf\,\{210, 245\} = 35$$

$$lcm = \frac{210 \times 245}{35}$$

$$= 1470$$

$\therefore$ *lcm* {210, 245} = 1470

(*Using a calculator*)

3.4 Exercise.

Question 1.

Find the *hcf* and *lcm* of 275 and 297

Question 2.

Find;

- **(i)** *hcf* {220, 462}
- **(ii)** *lcm* {220, 462}

Question 3.

(i) Write 336 as a product of primes.

(ii) Write 630 as a product of primes.

(iii) Calculate *hcf* {336, 630}.

(iv) Calcultae *lcm* {336, 630}.

Question 4.

(i) Find the *lcm* of 9 and 6.
(HINT : Work out the *hcf* first)

(ii) Find the *lcm* of 20 and 12.

(iii) Find the *lcm* of 18 and 12.

(iv) Find the *lcm* of 24 and 18.

Question 5.

(i) Write 875 as a product of primes.

(ii) Write 910 as a product of primes.

(iii) Calculate *hcf* {875, 910}.

(iv) Calcultae *lcm* {875, 910}.

Question 6.

(i) Find the *lcm* of 7 and 11.
(Hint : Work out the *hcf* first)

(ii) Find the *lcm* of 5 and 13.

(iii) Two prime numbers are p and q.
What is *lcm* $\{p, q\}$?

Question 7.

(i) Find the *lcm* of 4 and 16.

(ii) Find the *lcm* of 5 and 25.

(iii) Two numbers are s and s^2 .
What is *lcm* $\{s,\ s^2\}$?

Question 8.

(i) Find the *lcm* of 2 and 3.

(ii) Find the *lcm* of 3 and 4.

(iii) Two consecutive integers are a and b.
What is the *lcm* $\{a, b\}$?

Question 9.
Find the *lcm* of 525 and 2205.

Question 10.

A Table of Multiples

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216

Using the table of multiples, or otherwise, write down;

(i) *lcm*{2, 5}

(ii) *lcm* {3, 7}

(iii) *lcm* {8, 12}

(iv) *lcm* {4, 9}

Which of **(i), (ii), (iii)** and **(iv)** disproves the following theories;

THEORY A : The *lcm* of two numbers, *x* and *y*, is **always** *x* times *y*.

THEORY B : The *lcm* of two numbers, *x* and *y*, is *x* times *y* **only** if *x* and *y* are prime numbers.

Write down a correct theory that tells you when the *lcm* of two numbers, *x* and *y* is *x* times *y*.

Chapter 4

Number Theory : Year 9

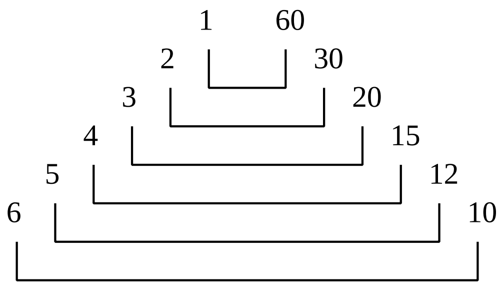
Calculator needed

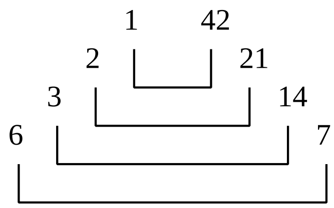
4.1 Factors and Multiples

4.1.1 Example involving factors

The factors of 60 are the whole numbers that divide into 60 without remainder.
The factors of 42 are the whole numbers that divide into 42 without remainder.

There is a good way of setting this out that lessens the chances of missing any **<u>factors</u>**:





This gives us a second method of finding an *hcf*, a *highest common* ***<u>factor</u>***

To find *hcf* {60, 42} look at both factor pyramids.
Which is the largest (the **<u>highest</u>**) number that is **<u>common</u>** to both **<u>factor</u>** pyramids ?

☞

4.1.2 Example involving Multiples

The multiples of 4 are the numbers in the 4-times **<u>multipl</u>**ication table:
4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, ...

The multiples of 21 are the numbers in the 21-times **<u>multipl</u>**ication table:
21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252, 273, 294, ...

This gives us a **second method** of finding a *lcm*, a *lowest commom* ***<u>multiple</u>***

To find *lcm* {4, 21} look through the mulpile lists for both 4 and 21.
Which is the first (the **<u>lowest</u>**) number that is **<u>common</u>** to both **<u>multiple</u>** lists ?

☞

What is the disadvantage of this method ?

☞

4.2 Exercise

Question 1

(i) Draw a factor pyramid for 24.

(ii) Draw a factor pyramid for 40.

(iii) What is the highest common factor of 24 and 40 ?

(iv) Draw a circle around *hcf* {24,40} on each of your factor pyramids.

(v) What is the highest common **PRIME** factor of 24 and 40 ?

(vi) List **ALL** the common factors of 24 and 40.

Question 2

(i) Write out a list containing, in order, the first twelve multiples of 9.

(ii) Write out a list containing, in order, the first twelve multiples of 15.

(iii) What is the lowest common multiple of 9 and 15 ?

(iv) Draw a circle around *lcm* {9,15} on each of your multpile lists.

(v) What is the second lowest common multiple of 9 and 15 ?

(vi) What do you think the third lowest common multpile of 9 and 15 will be ?

Question 3

(i) Write 195 as a product of primes.

(ii) Hence, or otherwise, draw the factor pyramid for 195

Question 4.

(i) Draw a factor pyramid for 54.

(ii) Draw a factor pyramid for 36.

(iii) Draw a factor pyramid for 64.

(iv) Use your factor pyramids to state *hcf* { 54, 36, 64 }

(v) Circle the hcf on each of your factor pyramids.

Question 5

(i) Write out a list containing, in order, multiples of 6 less than 100.

(ii) Write out a list containing, in order, multiples of 14 less than 100.

(iii) Write out a list containing, in order, multiples of 12 less than 100.

(iv) What is the lowest common multiple of 6 and 14 and 12 ?

(v) Draw a circle around *lcm* {6,14, 12} on each of your multpile lists.

Question 6
Here is another way of finding the lcm of three numbers.
Suppose we wish to find lcm { 10, 25, 30}

(i) Pick any two numbers from the three, say 10 and 25.
Write down *lcm* { 10, 25 }

(ii) Pick any two different numbers from the three, say 10 and 30.
Write down *lcm* [10, 30 }

(iii) Now find the lcm of your answers to part **(i)** and part **(ii)**.
This is also the *lcm* of all three numbers.

Question 7
Use the method of question 6 to find lcm { 14, 32, 40 }

Chapter 5

Number Theory : Year 9

5.1 REVISION

Calculator needed

Question 1.

Write down the *hcf* and the *lcm* requested;

(i) *hcf* {12, 20} = *lcm* {12, 20} =

(ii) *hcf* {12, 13} = *lcm* {12, 13} =

(iii) *hcf* {66, 88} = *lcm* {66, 88} =

(iv) *hcf* {30, 45} = *lcm* {30, 45} =

[8 marks]

Question 2.

Is 637 a prime number ?
Explain your answer.

[3 marks]

Question 3.

(i) List the first ten multiples of 12.

(ii) List the first ten multiples of 15.

(iii) Which is the first multiple that is common to both your part **(i)** and part **(ii)** lists ?

(iv) Is this number the *hcf* or the *lcm* ?

[6 marks]

Question 4.
Write down a prime number that is even or, if there is not one, write NONE.

[1 mark]

Question 5.
(i) Write down all the factors of 24.

(ii) Write down all the prime factors of 24.

(iii) Write down the highest prime factor of 24.

[4 marks]

Question 6.
Which of the following pairs of numbers are co-prime ?
(*coprime* means *have no common factors, except 1*)

(i) 4 and 9 **(ii)** 4 and 12 **(iii)** 4 and 13

[3 marks]

Question 7.
(i) Write down three square numbers.

(ii) Can a square number be prime ?

[4 marks]

Question 8.

The **FUNDAMENTAL THEOREM OF ARITHMETIC** says that any number which is not prime can be written as a product of primes in, essentially, a unique way.

(i) What does the word unique mean ?

[1 mark]

(ii) Write 108 as a product of primes.

[4 marks]

(iii) Write 3575 as a product of primes.

[4 marks]

Question 9.
Find the *highest common factor* of 350 and 375

[5 marks]

Question 10.
Find the *lowest common multiple* of 90 and 140

[5 marks]

Question 11.
The Goldback conjecture states that any even number greater than four can be written as the sum of two prime numbers.
Write the following even numbers as sums of two primes.

(a) 30

(b) 22

(c) 24

(d) 52

It is not known if the Goldback conjecture is true or not.

[4 marks]

Question 12.
Find the *highest common factor* of 96, 60 and 216.

[6 marks]

Question 13.
Find the *lowest common multiple* of 6, 8 and 14.

[6 marks]

Question 14.

(a) What is 19 times 11 ?

[1 mark]

Prof Eureka Fish claims to have a formula that generates prime numbers.
Here is his method;

Take the first *x* primes.
Multiply them all together.
Subtract 1.
The answer is a new prime.

(b) **(i)** Work out $2 \times 3 - 1$
Is it prime ?

[1 mark]

(ii) Work out $2 \times 3 \times 5 - 1$
Is it prime ?

[1 mark]

(iii) Work out $2 \times 3 \times 5 \times 7 - 1$
Is it prime ?

[1 mark]

(iv) Do you think the Prof is correct or not ?
Justify your answer.

[2 marks]

5.1 Solutions

Answer 1.

(i)	4	$\frac{12 \times 20}{4} = 60$
(ii)	1	$\frac{12 \times 13}{1} = 156$
(iii)	22	$\frac{66 \times 88}{22} = 264$
(iv)	15	$\frac{30 \times 45}{15} = 90$

Answer 2.

A prime number, *p*, has exactly two factors, 1 and *p*
637 is divisible by, for example, 7 as well as 1 and 637.
$\therefore$ 637 has more than two factors.
$\therefore$ 637 is not prime.

Answer 3.

(i) 12, 24, 36, 48, 60, 72, 84, 96, 108, 120
(ii) 15, 30, 45, 60, 75, 90, 105, 120, 135, 150
(iii) 60
(iv) *lcm*

Answer 4.

2

Answer 5.

(i) 1, 2, 3, 4, 6, 8, 12, 24
(ii) 2, 3
(iii) 3

Answer 6.

(i)	Yes.	4 and 9 are coprime
(ii)	No.	Factor of 2 and 4 in common.
(iii)	Yes.	4 and 13 are comprime.

Answer 7.

(i) 1, 4, 9 etc
(ii) No.

Answer 8.

(i) One of a kind, only one etc

(ii) $108 = 2 \times 2 \times 3 \times 3 \times 3$

(iii) $3575 = 5 \times 5 \times 11 \times 13$

Answer 9.

$350 = 2 \times 5 \times 5 \times 7$

$375 = 3 \times 5 \times 5 \times 5$ $\quad$ $hcf = 25$

Answer 10.

$90 = 2 \times 3 \times 3 \times 5$

$140 = 2 \times 2 \times 5 \times 7$ $\quad$ $hcf = 10$ $\quad$ $lcm = \frac{90 \times 140}{10} = 1260$

Answer 11.

(a) $30 = 7 + 23$
$= 11 + 19$
$= 13 + 17$

(b) $22 = 3 + 19$
$= 5 + 17$
$= 11 + 11$

(c) $24 = 5 + 19$
$= 7 + 17$
$= 11 + 13$

(d) $52 = 5 + 47$
$= 11 + 41$
$= 23 + 29$

Answer 12.

$96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$

$60 = 2 \times 2 \times 3 \times 5$

$216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$ $\quad$ $hcf = 12$

Answer 13.

$6 = 2 \times 3$ $\quad$ $hcf\{6, 8\} = 2 \Rightarrow lcm\{6, 8\} = 24$

$8 = 2 \times 2 \times 2$ $\quad$ $hcf\{6, 14\} = 2 \Rightarrow lcm\{6, 14\} = 42$

$14 = 2 \times 7$ $\quad$ *hcf* of answers is 6

lcm of answers is 168

i.e. $lcm\{6,8, 14\} = 168$

Answer 14.

(a) 209

(b) **(i)** 5 Yes, prime

(ii) 29 Yes, prime

(iii) 209 No, NOT prime

(iv) $\therefore$ Prof Eureka Fish is wrong.

Chapter 6

Number Theory : Year 9

6.1 TEST

Calculator needed

Question 1.

Write down the *hcf* and the *lcm* requested;

(i) *hcf* {14, 21} = *lcm* {14, 21} =

(ii) *hcf* {11, 12} = *lcm* {11, 12} =

(iii) *hcf* {66, 99} = *lcm* {66, 99} =

(iv) *hcf* {40, 48} = *lcm* {40, 48} =

[8 marks]

Question 2.

Is 1001 a prime number ?
Explain your answer.

[3 marks]

Question 3.

(i) List the first ten multiples of 45.

(ii) List the first ten multiples of 54.

(iii) Which is the first multiple that is common to both your part **(i)** and part **(ii)** lists ?

(iv) Is this number the *hcf* or the *lcm* ?

[6 marks]

Question 4.
Write down a prime number that is divisible by 3 or, if there is not one, write NONE.

[1 mark]

Question 5.
(i) Write down all the factors of 90.

(ii) Write down all the prime factors of 90.

(iii) Write down the highest prime factor of 90.

[4 marks]

Question 6.
Which of the following pairs of numbers are co-prime ?
(*coprime* means *have no common factors, except 1*)

(i) 9 and 25 **(ii)** 9 and 12 **(iii)** 9 and 13

[3 marks]

Question 7.
(i) Write down three square numbers.

(ii) Can a square number be a factor of 125 ?

[4 marks]

Question 8.

The **FUNDAMENTAL THEOREM OF ARITHMETIC** says that any number which is not prime can be written as a product of primes in, essentially, a unique way.

(i) What does the word product mean ?

[1 mark]

(ii) Write 180 as a product of primes.

[4 marks]

(iii) Write 2695 as a product of primes.

[4 marks]

Question 9.
Find the *highest common factor* of 210 and 525

[5 marks]

Question 10.
Find the *lowest common multiple* of 105 and 150

[5 marks]

Question 11.
A number is said to be **PERFECT** if that number is equal to half the sum of its factors.
For example, $6 = \frac{1}{2}(1 + 2 + 3 + 6)$ $\therefore$ 6 is perfect.

Which of the following numbers are perfect ?
In each case evidence your answer with working.

(i) 24 **(ii)** 28 **(iii)** 31

[6 marks]

Question 12.
Find the *highest common factor* of 126, 196 and 350.

[6 marks]

Question 13.
Find the *lowest common multiple* of 16, 28 and 40.

[6 marks]

Question 14.

Look at the following table:

	1	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	...			

(i) Continue this table for five more lines.

[1 mark]

(ii) Circle the prime numbers.

[2 marks]

(iii) Explain what you see.

[6 marks]

Chapter 7

Number Theory : Year 9

7.1 Trapping the primes (Extension Material)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

THEOREM :
No number in the column headed by the "4" can be prime, no matter how long the table.

PROOF :
All numbers in the column headed by the "4" are of the type:

$$y = 10n + 4, \qquad for\ n = 0, 1, 2, 3, \ldots$$

$$\Leftrightarrow y = 2(5n + 2)$$

This shows that these numbers are always divisible by 2, and so are never prime.

□

7.2 Exercise

Question 1.

(a) Write down four numbers that are in the column;

$$y = 10n + 6, \qquad for \quad n = 0, 1, 2, 3, \ldots$$

(b) Rewrite $y = 10n + 6$ in a way that shows that no prime numbers are in this column.

Question 2.

(a) Write down two prime numbers that are in the column;

$$y = 10n + 1, \qquad for \quad n = 0, 1, 2, 3, \ldots$$

(b) Write down two numbers, NOT PRIME, that are in the column;

$$y = 10n + 1, \qquad for \ n = 0, 1, 2, 3, \ldots$$

This shows that prime numbers can be of the type $y = 10n + 1$ but that not every number of this type is prime.

Question 3.

(a) Write down six numbers described by;

$$y = 10n + 2, \qquad for \ n = 1, 2, 3, \ldots$$

(b) Only one number in this column is prime.
Which number ?

(c) How did I exclude this number in the formula of part **(a)** ?

Question 4.

(a) Write down a formula for the numbers in the column headed by "5". Do not allow your formula to generate the number 5. (As 5's prime).

(b) Show that all of these numbers are divisible by 5 by rewriting your part (a) formula with brackets.

Question 5.

In the 'six wide' and 'seven wide' table, shade all numbers that are not prime.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63
64	65	66	67	68	69	70
71	72	73	74	75	76	77
78	79	80	81	82	83	84
85	86	87	88	89	90	91
92	93	94	95	96	97	98
99	100	101	102	103	104	105
106	107	108	109	110	111	112
113	114	115	116	117	118	119

Which table is the better trapper of primes ?

Chapter 8

8.1 'Not Prime' Proofs (Extension Material)

Chapter 3 concluded by investigating where the primes where located in grids of numbers six and seven columns wide. Here is a recap of what was found;

1	**2**	**3**	4	**5**	6
7	8	9	10	**11**	12
13	14	15	16	**17**	18
19	20	21	22	**23**	24
25	26	27	28	**29**	30
31	32	33	34	35	36
37	38	39	40	**41**	42
43	44	45	46	**47**	48
49	50	51	52	**53**	54
55	56	57	58	**59**	60
61	62	63	64	65	66
67	68	69	70	**71**	72
73	74	75	76	77	78
79	80	81	82	**83**	84
85	86	87	88	**89**	90
91	92	93	94	95	96
97	98	99	100	**101**	102

1	**2**	**3**	4	**5**	6	**7**
8	9	10	**11**	12	**13**	14
15	16	**17**	18	**19**	20	21
22	**23**	24	25	26	27	28
29	30	**31**	32	33	34	35
36	**37**	38	39	40	**41**	42
43	44	45	46	**47**	48	49
50	51	52	**53**	54	55	56
57	58	**59**	60	**61**	62	63
64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77
78	**79**	80	81	82	**83**	84
85	86	87	88	**89**	90	91
92	93	94	95	96	**97**	98
99	100	**101**	102	**103**	104	105
106	**107**	108	**109**	110	111	112
113	114	115	116	117	118	119

The two tables are very different.
The 'six' table traps the primes in just two of its six columns if we exclude the top row.

For *n* = 1, 2, 3, ...

'six' table algebraically

$$Column\ 1 \ :\ y = 6n + 1 \qquad \therefore\ Primes\ !$$

$$Column\ 2 \ :\ y = 6n + 2 = 2(3n + 1) \qquad \therefore\ No\ primes$$

$$Column\ 3 \ :\ y = 6n + 3 = 3(2n + 1) \qquad \therefore\ No\ primes$$

$$Column\ 4 \ :\ y = 6n + 4 = 2(3n + 2) \qquad \therefore\ No\ primes$$

$$Column\ 5 \ :\ y = 6n + 5 \qquad \therefore\ Primes\ !$$

$$Column\ 6 \ :\ y = 6n + 6 = 6(\ n + 1) \qquad \therefore\ No\ primes$$

The algebraic analysis is less effort than shading a table !

The 'seven' table only succeeds in not having primes in it's final column.
(Again, this is excluding the top row by not allowing $n = 0$ into the formulae)

For *n* = 1, 2, 3, ...

'seven' table algebraically

$$Column\ 1 \ : \ y = 7n + 1 \qquad \therefore Primes!$$

$$Column\ 2 \ : \ y = 7n + 2 \qquad \therefore Primes!$$

$$Column\ 3 \ : \ y = 7n + 3 \qquad \therefore Primes!$$

$$Column\ 4 \ : \ y = 7n + 4 \qquad \therefore Primes!$$

$$Column\ 5 \ : \ y = 7n + 5 \qquad \therefore Primes!$$

$$Column\ 6 \ : \ y = 7n + 6 \qquad \therefore Primes!$$

$$Column\ 7 \ : \ y = 7n + 7 = 7(n + 1) \qquad \therefore No\ primes$$

The algebra confirms that the 'seven' table didn't really trap the primes at all.

8.2 The 'eight' table algebraically.

Analyse the 'eight' table using algebra.

For *n* = 1, 2, 3, ...

'eight' table algebraically

$$Column\ 1 \ : \ y = 8n + 1 \qquad \therefore Primes!$$

$$Column\ 2 \ : \ y =$$

$$Column\ 3 \ : \ y =$$

$$Column\ 4 \ : \ y =$$

$$Column\ 5 \ : \ y =$$

$$Column\ 6 \ : \ y =$$

$$Column\ 7 \ : \ y =$$

$$Column\ 8 \ : \ y =$$

8.3 The 'eight' table with shading.

1	**2**	**3**	4	**5**	6	**7**	8
9	10	**11**	12	**13**	14	15	16
17	18	**19**	20	21	22	**23**	24
25	26	27	28	**29**	30	**31**	32
33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48
49	50	51	52	**53**	54	55	56
57	58	**59**	60	**61**	62	63	64
65	66	**67**	68	69	70	**71**	72
73	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88
89	90	91	92	93	94	95	96
97	98	99	100	**101**	102	**103**	104

8.4 Exercise.

Question 1.

Analyse the 'nine' table using algebra, *for* $n = 1, 2, 3, ...$

$Column\ 1 \ : \ y \ = \ 9n \ + \ 1$ $\qquad\qquad \therefore \ Primes\,!$

$Column\ 2 \ : \ y \ =$

$Column\ 3 \ : \ y \ =$

$Column\ 4 \ : \ y \ =$

$Column\ 5 \ : \ y \ =$

$Column\ 6 \ : \ y \ =$

$Column\ 7 \ : \ y \ =$

$Column\ 8 \ : \ y \ =$

$Column\ 9 \ : \ y \ =$

Question 2.

(a) In the 'sixteen' table with top row removed, write down the first six numbers in the "3" column which has the formula;

$$y = 16n + 3 \qquad for \quad n = 1, \; 2, \; 3, \; ...$$

(b) This "3" column contains numbers that are composite.
(*A composite number has more than two factors*)
Does it also contain prime numbers ?

Question 3.

(a) In the 'eighteen' table with top row removed, write down the first six numbers in the "12" column which has the formula;

$$y = 18n + 12 \qquad for \quad n = 1, \; 2, \; 3, \; ...$$

(b) The formula can be rewritten in the following way:

$$y = 2(9n + 6) \qquad for \quad n = 1, \; 2, \; 3, \; ...$$

What does this show that numbers in the "12" column are divisible by ?

(c) The formula can be rewritten in the following way:

$$y = 3(6n + 4) \qquad for \quad n = 1, \; 2, \; 3, \; ...$$

What does this show that numbers in the "12" column are divisible by ?

(d) The formula can be rewritten in the following way:

$$y = 6(3n + 2) \qquad for \quad n = 1, \; 2, \; 3, \; ...$$

What ***three*** numbers does this show that numbers in the "12" column are divisible by ?

(e) What is *hcf* {18, 12} ?

Question 4.

(a) In the 'thirty-six' table with top row removed, write down the first six numbers in the "24" column which has the formula;

$$y = 36n + 24 \qquad for \quad n = 1, 2, 3, \ldots$$

(b) What is hcf{36,24} ?

(c) Rewrite the part (a) formula in a way that shows all numbers in the "24" column divide by 2, 3, 4, 6 and 12 and so are composite.

Question 5.

In the 'fifteen' table the formula for the column headed "10" is of the form;

$$y = an + b \qquad for \quad n = 0, 1, 2, 3, \ldots$$

(i) State the values of a and b.

(ii) Using the part **(i)** values of a and b, what is $hcf\,\{a, b\}$?

(iii) Using the part **(ii)** value for $hcf\,\{a, b\}$ show that every number in the "10" column can be divided by $hcf\,\{a, b\}$ and so is NOT prime.

Question 6.

(i) Find hcf {70, 66}.

(ii) In the 'seventy' table, will column "66" contain any prime numbers ? Justify your answer by considering the formula;

$$y = 70n + 66 \qquad for \quad n = 1, 2, 3, \ldots$$

Question 7.

The 'six' table was particularly good at trapping the primes in just a few columns.
Prof P Stalker claims that this is because the first two primes are 2, 3 and $2 \times 3 = 6$.
The Prof's arguement suggest that the next good prime trapping table will be
the 'thirty' table because the first three primes are 2, 3, 5 and $2 \times 3 \times 5 = 30$.

Analyse the 'thirty' table using algebra, *for* $n = 1, 2, 3, \ldots$

$Column\ 1\ :\ y\ =\ 30n\ +\ 1$ $\qquad\qquad \therefore\ Primes\ !$

$Column\ 2\ :\ y\ =\ 30n\ +\ 2\ =\ 2\,(15n\ +\ 1)$ $\qquad\qquad \therefore\ No\ primes.$

$Column\ 3\ :\ y\ =$

$Column\ 4\ :\ y\ =$

$Column\ 5\ :\ y\ =$

$Column\ 6\ :\ y\ =$

$Column\ 7\ :\ y\ =$

$Column\ 8\ :\ y\ =$

$Column\ 9\ :\ y\ =$

$Column\ 10\ :\ y\ =$

$Column\ 11\ :\ y\ =$

$Column\ 12\ :\ y\ =$

$Column\ 13\ :\ y\ =$

$Column\ 14\ :\ y\ =$

$Column\,15 \;:\; y =$

$Column\,16 \;:\; y =$

$Column\,17 \;:\; y =$

$Column\,18 \;:\; y =$

$Column\,19 \;:\; y =$

$Column\,20 \;:\; y =$

$Column\,21 \;:\; y =$

$Column\,22 \;:\; y =$

$Column\,23 \;:\; y =$

$Column\,24 \;:\; y =$

$Column\,25 \;:\; y =$

$Column\,26 \;:\; y =$

$Column\,27 \;:\; y =$

$Column\,28 \;:\; y =$

$Column\,29 \;:\; y =$

$Column\,30 \;:\; y =$

The 'Thirty' table by Prof P Stalker.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210

Question 7 (Continued).
The 'six' table trapped the primes in two of its six columns.

(i) Write $\frac{2}{6}$ as a percentage.

(ii) What percentage of the available columns in the 'thirty' table are the primes trapped in ?

(iii) Is the 'six' table or the 'thirty' table a better prime trapper ? Explain your answer.

(iv) If the Prof's theory is correct, what would be the next width of table worth looking at ?

(v) What percentage of the available columns would the primes be trapped in in this next table ?