A-Level Pure Mathematics

Core 1

Core 1 is a **NON CALCULATOR** module

Topic N° 1

Surds & Indices

$\sqrt{\;S \cdot U \cdot R \cdot D \cdot S\;}$

Surds & Indices

Core 1 is a **NON CALCULATOR** module

Chapter 1

Algebra : Core 1

1.1 Square Roots without a calculator

Example : Find $\sqrt{2704}$

1.2 Exercise

Question 1

Determine,

(i) $\sqrt{3969}$

(ii) $\sqrt{5625}$

1.3 Square Free

Any number which is not prime can be written as a unique product of primes.
For example,

$$120 = 2^3 \times 3 \times 5$$

Mathematicians talk of the *decomposition* of 120 into a product of primes.

There is another decomposition of 120 that is useful. It revolves around identifying the biggest square number that will divide into 120 exactly.

Reminder: Square Numbers = { 1, 4, 9, 16, 25, 36, ... }

As 4 is the biggest square number that divides into 120 we can write,

$$120 = 4 \times 30$$

Notice that no square number, other than 1, will divide into 30.
Thus 30 is termed square-free or, more succinctly, □ Free.

In summary, our new decomposition takes an integer that is not □ Free and expresses it as a square number multiplied by a square free number.
i.e.

$$Non\,\square = \square \times \square\,Free$$

Example : Write $\sqrt{120}$ in the form $a\sqrt{p}$
where a & p are integers
and p is □ FREE.

1.4 Exercise

Question 1

Write each of the following in the form $a\sqrt{p}$

where a & p are integers

and p is □ FREE.

(i) $\sqrt{8}$

(ii) $\sqrt{27}$

(iii) $\sqrt{48}$

(iv) $\sqrt{98}$

(v) $\sqrt{56}$

(vi) $\sqrt{242}$

(vii) $\sqrt{320}$

Question 2

Write each of the following in the form $a\sqrt{p}$

where a & p are integers

and p is □ FREE.

(i) $\sqrt{504}$

(ii) $\sqrt{1452}$

(iii) $\sqrt{6750}$

(iv) $\sqrt{5346}$

Question 3

What is the cube root of 1728 ?

$i.e.\quad \sqrt[3]{1728}$

1.5 Answers

1.5.1 Solution (1.1 Example)

2	2704
2	1352
2	676
2	338
13	169
13	13
	1

Method 1 : Less Mathematical

$$\therefore \sqrt{2704} = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{13} \times \sqrt{13}$$

$$\sqrt{2704} = (\sqrt{2} \times \sqrt{2}) \times (\sqrt{2} \times \sqrt{2}) \times (\sqrt{13} \times \sqrt{13})$$

$$\sqrt{2704} = (2) \times (2) \times (13)$$

$$= 4 \times 13$$

$$= 52$$

Method 2 : More Mathematical

$$2704 = 2^4 \times 13^2$$

$$\therefore \sqrt{2704} = \sqrt{2^4 \times 13^2}$$

$$= (2^4 \times 13^2)^{\frac{1}{2}}$$

$$= 2^2 \times 13$$

$$= 52$$

1.5.2 Solution (1.2 Exercise)

Answer 1

(i) 63 **(ii)** 75

1.5.3 Solution (1.3 Example)

Method 1 : Less Mathematical

$$\sqrt{120} = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{3} \times \sqrt{5}$$
$$= (\sqrt{2} \times \sqrt{2}) \times \sqrt{2} \times \sqrt{3} \times \sqrt{5}$$
$$= (2) \times \sqrt{2} \times \sqrt{3} \times \sqrt{5}$$
$$= 2\sqrt{30}$$

Method 2 : More Mathematical

$$\sqrt{120} = \sqrt{4} \times \sqrt{30}$$
$$= 2\sqrt{30}$$

1.5.4 Solutions (1.4 Exercise)

Answer 1

(i) $8 = 4 \times 2$
$\therefore \sqrt{8} = 2\sqrt{2}$

(ii) $27 = 9 \times 3$
$\therefore \sqrt{27} = 3\sqrt{3}$

(iii) $48 = 16 \times 3$
$\therefore \sqrt{48} = 4\sqrt{3}$

(iv) $98 = 49 \times 2$
$\therefore \sqrt{98} = 7\sqrt{2}$

(v) $56 = 4 \times 14$
$\therefore \sqrt{56} = 2\sqrt{14}$

(vi) $242 = 121 \times 2$
$\therefore \sqrt{242} = 11\sqrt{2}$

(vii) $320 = 64 \times 5$
$\therefore \sqrt{320} = 8\sqrt{5}$

Answer 2

(i) $504 = 36 \times 14$
$\therefore \sqrt{504} = 6\sqrt{14}$

(ii) $1452 = 484 \times 3$
$\therefore \sqrt{1452} = 22\sqrt{3}$

(iii) $6750 = 225 \times 30$
$\therefore \sqrt{6750} = 15\sqrt{30}$

(iv) $5346 = 81 \times 66$
$\therefore \sqrt{5346} = 9\sqrt{66}$

Answer 3

12

1.6 Homework

Question 1

Use the fact that $405 = 3^4 \times 5$ to write $\sqrt{405}$ in the form $a\sqrt{p}$

where a and p are integers

and p is ☐ FREE. (And also, in this case, prime)

Question 2

Write each of the following in the form $a\sqrt{p}$

where a & p are integers

and p is ☐FREE. (And also, in these cases, prime)

(i) $\sqrt{44}$

(ii) $\sqrt{50}$

(iii) $\sqrt{32}$

(iv) $\sqrt{99}$

(v) $\sqrt{200}$

(vi) $\sqrt{162}$

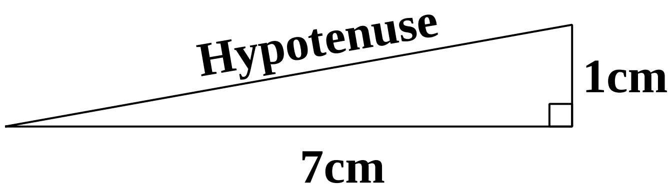
Question 3

Find the **exact** length of the hypotenuse of a right-angled △ of base 7 cm and height 1 cm.

Write your answer in the form $a\sqrt{p}$

where a & p are integers

and p is ☐FREE. (and also, in this case, prime)



Question 4

Find the *exact* length of the hypotenuse of a right angled △ of base 6 cm and height 2 cm.

Write your answer in the form $a\sqrt{f}$

where a & f are integers

and f is ☐ FREE. (Note: f is not prime)

Question 5

Find the *exact* length of the hypotenuse of a right-angled △of base 10 cm and height 4 cm.

Write your answer in the form $a\sqrt{p}$

where a & p are integers

and p is ☐ FREE. (and also, in this case, prime)

Question 6

Find the **exact** length of the hypotenuse of a right-angled △ of base 13 cm and height 9 cm.

Write your answer in the form $a\sqrt{f}$

where a & f are integers

and f is ☐ FREE. (Note: f is not prime)

Question 7

Find the exact length of the hypotenuse of a right-angled △ of base 15 cm and height 9 cm.

Write your answer in the form $a\sqrt{f}$

where a & f are integers

and f is □ FREE. (Note: f is not prime)

DECLARATION:

I, ________________________________, being of sound body and brain,
do declare that I have not used a calculator in answering any of these questions.

Nor did I look to the left nor the right, at my neighbours answers.
(which would have been wrong anyway)

Furthermore, I love maths.

Signed: ____

1.7 Answers

1.7.1 Solutions (1.6 Homework)

Answer 1

$9\sqrt{5}$

Answer 2

(i)	$2\sqrt{11}$	**(ii)**	$5\sqrt{2}$
(iii)	$4\sqrt{2}$	**(iv)**	$3\sqrt{11}$
(v)	$10\sqrt{2}$	**(vi)**	$9\sqrt{2}$

Answer 3

$5\sqrt{2}$

Answer 4

$2\sqrt{10}$

Answer 5

$2\sqrt{29}$

Answer 6

$5\sqrt{10}$

Answer 7

$3\sqrt{34}$

Chapter 2

2.1 SURD Manipulations

There are benefits in 'using your wits' to try and spot 'short cuts' and 'clever moves' to avoid long-winded and pedantic solutions.

However, if you need to put in lots of steps to be certain of getting a question right, put them in - it's often quicker to scribble down a feqw extra lines rather than agonise over a tricky step in your mind.

Also keep in mind that there are often a couple of different methods that could beemployed - use what you are comfortable with, which may differ from your neighbour has done.

Example

Calculate $2\sqrt{15} \times 4\sqrt{10}$

Write your answer in the form $a\sqrt{b}$

where a and b are integers

and b is as small as possible.

2.2 Exercise

Question 1

Calculate each of the following, writing your answers in the form $a\sqrt{b}$

where a & b are integers

and b is as small as possible.

(i) $3\sqrt{6} \times 7\sqrt{21}$

(ii) $5\sqrt{14} \times 2\sqrt{10}$

(iii) $10\sqrt{22} \times 4\sqrt{6}$

(iv) $3\sqrt{10} \times 4\sqrt{55}$

Question 2

Simplify:

(i)	$\sqrt{12}$	**(ii)**	$\sqrt{20}$	**(iii)**	$\sqrt{8}$
(iv)	$\sqrt{45}$	**(v)**	$\sqrt{48}$	**(vi)**	$\sqrt{242}$
(vii)	$\sqrt{75}$	**(viii)**	$\sqrt{162}$	**(ix)**	$\sqrt{147}$
(x)	$\sqrt{125}$	**(xi)**	$\sqrt{567}$	**(xii)**	$\sqrt{112}$

Question 3

Simplify:

(i)	$5\sqrt{18}$	**(ii)**	$2\sqrt{300}$	**(iii)**	$5\sqrt{54}$
(iv)	$3\sqrt{80}$	**(v)**	$4\sqrt{175}$	**(vi)**	$2\sqrt{245}$

Question 4

Simplify:

(i) $\frac{\sqrt{44}}{2}$ **(ii)** $\frac{\sqrt{24}}{2}$ **(iii)** $\frac{\sqrt{200}}{5}$

(iv) $\frac{\sqrt{243}}{3}$ **(v)** $\frac{\sqrt{288}}{4}$ **(vi)** $\frac{\sqrt{450}}{3}$

Question 5

Simplify:

(i) $\frac{\sqrt{98}}{\sqrt{2}}$ **(ii)** $\frac{\sqrt{500}}{\sqrt{5}}$ **(iii)** $\frac{\sqrt{63}}{\sqrt{7}}$

Question 6

Simplify:

(i) $\frac{\sqrt{10}}{\sqrt{5}}$ **(ii)** $\frac{\sqrt{22}}{\sqrt{11}}$ **(iii)** $\frac{\sqrt{56}}{\sqrt{7}}$

Question 7

Simplify:

(i) $3\sqrt{75} + 2\sqrt{12}$

(ii) $2\sqrt{18} + \sqrt{200} - \sqrt{72}$

(iii) $5\sqrt{20} + 3\sqrt{45} - 4\sqrt{80}$

(iv) $6\sqrt{6} - \sqrt{24} + 3\sqrt{294}$

Question 8

Simplify:

(i) $5\sqrt{63}$

(ii) $7\sqrt{200}$

(iii) $10\sqrt{216}$

(iv) $2\sqrt{90}$

(v) $11\sqrt{375}$

(vi) $12\sqrt{288}$

Question 9

Write each of the following in the form $a + b\sqrt{c}$ for integer a, b and c.
Furthermore, c is to be square free.
[Your answers to **Question 8** will be helpful...]

(i) $\dfrac{18 + 5\sqrt{63}}{3}$

(ii) $\dfrac{14 + 7\sqrt{200}}{14}$

(iii) $\dfrac{45 + 10\sqrt{216}}{15}$

(iv) $\dfrac{12 + 2\sqrt{90}}{3}$

(v) $\dfrac{30 - 11\sqrt{375}}{5}$

(vi) $\dfrac{36 + 12\sqrt{288}}{9}$

Question 10

Write in the form $a + b\sqrt{c}$ where a, b and c are integers.
Furthermore, c, is to be □ Free.

(i) $\dfrac{200 + 3\sqrt{1000}}{5}$

(ii) $\dfrac{-64 + 16\sqrt{88}}{32}$

(iii) $\dfrac{2 + 3\sqrt{156}}{2}$

2.3 Answers

2.3.1 Solutions (2.1 Example)

Method 1 : Less Mathematical

$$2\sqrt{15} \times 4\sqrt{10}$$
$$= 2 \times \sqrt{3} \times \sqrt{5} \times 4 \times \sqrt{2} \times \sqrt{5}$$
$$= 2 \times 4 \times (\sqrt{5} \times \sqrt{5}) \times \sqrt{3} \times \sqrt{2}$$
$$= 8 \times 5 \times \sqrt{3} \times \sqrt{2}$$
$$= 40\sqrt{6}$$

Method 2 : More Mathematical

$$2\sqrt{15} \times 4\sqrt{10}$$
$$= 8 \times \sqrt{150}$$
$$= 8 \times \sqrt{25 \times 6}$$
$$= 8 \times 5 \times \sqrt{6}$$
$$= 40\sqrt{6}$$

2.3.2 Solutions (2.2 Exercise)

Answer 1

(i) $63\sqrt{14}$ **(ii)** $20\sqrt{35}$

(iii) $80\sqrt{33}$ **(iv)** $60\sqrt{22}$

Answer 2

(i) $2\sqrt{3}$ **(ii)** $2\sqrt{5}$ **(iii)** $2\sqrt{2}$

(iv) $3\sqrt{5}$ **(v)** $4\sqrt{3}$ **(vi)** $11\sqrt{2}$

(vii) $5\sqrt{3}$ **(viii)** $9\sqrt{2}$ **(ix)** $7\sqrt{3}$

(x) $5\sqrt{5}$ **(xi)** $9\sqrt{7}$ **(xii)** $4\sqrt{7}$

Answer 3

(i) $15\sqrt{2}$ **(ii)** $20\sqrt{3}$ **(iii)** $15\sqrt{6}$

(iv) $12\sqrt{5}$ **(v)** $20\sqrt{7}$ **(vi)** $14\sqrt{5}$

Answer 4

(i) $\sqrt{11}$ **(ii)** $\sqrt{6}$ **(iii)** $2\sqrt{2}$

(iv) $3\sqrt{3}$ **(v)** $3\sqrt{2}$ **(vi)** $5\sqrt{2}$

Answer 5

(i) 7 **(ii)** 10 **(iii)** 3

Answer 6

(i)	$\sqrt{2}$	**(ii)**	$\sqrt{2}$	**(iii)**	$2\sqrt{2}$

Answer 7

(i) $19\sqrt{3}$
(ii) $10\sqrt{2}$
(iii) $3\sqrt{5}$
(iv) $25\sqrt{6}$

Answer 8

(i)	$15\sqrt{7}$	**(ii)**	$70\sqrt{2}$
(iii)	$60\sqrt{6}$	**(iv)**	$6\sqrt{10}$
(v)	$55\sqrt{15}$	**(vi)**	$144\sqrt{2}$

Answer 9

(i)	$6 + 5\sqrt{7}$	**(ii)**	$1 + 5\sqrt{2}$
(iii)	$3 + 4\sqrt{6}$	**(iv)**	$4 + 2\sqrt{10}$
(v)	$6 - 11\sqrt{15}$	**(vi)**	$4 + 16\sqrt{2}$

Answer 10

(i) $40 + 6\sqrt{10}$
(ii) $-2 + \sqrt{22}$
(iii) $1 + 3\sqrt{39}$

3.1 Easy "Rationalising the Denominator"

Mathematicians' dislike fractions which have a square root in the denominator.
There are standard techniques for manipulating such fractions to remove the offending square root from the denominator.
This may well result in a square root in the numerator, but this is considered fine !

Example:

Rationalise the denominator of $\frac{4\sqrt{3}}{\sqrt{5}}$

3.2 Exercise

Question 1.

Rationalise the denominators of the following fractions;

(i) $\frac{20}{\sqrt{5}}$ **(ii)** $\frac{28}{\sqrt{7}}$ **(iii)** $\frac{24\sqrt{3}}{\sqrt{2}}$

(iv) $\frac{12}{\sqrt{3}}$ **(v)** $\frac{5}{\sqrt{13}}$ **(vi)** $\frac{14\sqrt{3}}{\sqrt{2}}$

(vii) $\frac{55}{\sqrt{11}}$ **(viii)** $\frac{1}{\sqrt{2}}$ **(ix)** $\frac{15\sqrt{2}}{\sqrt{15}}$

Question 2.

Rationalise the denominators of the following fractions;

(i) $\dfrac{52}{3\sqrt{13}}$ **(ii)** $\dfrac{48}{5\sqrt{6}}$ **(iii)** $\dfrac{7}{3\sqrt{15}}$

(iv) $\dfrac{11}{12\sqrt{3}}$ **(v)** $\dfrac{6\sqrt{3}}{7\sqrt{2}}$ **(vi)** $\dfrac{3}{\sqrt{7}}$

(vii) $\dfrac{44}{5\sqrt{11}}$ **(viii)** $\dfrac{14}{\sqrt{2}}$ **(ix)** $\dfrac{28}{3\sqrt{14}}$

3.3 Answers.

3.3.1 Solutions (3.1 Introductory Example)

$$\textit{Multiple “top \& bottom” by } \sqrt{5}$$

$$= \frac{4\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{4\sqrt{15}}{5} \qquad (\textit{which is equivalent to } 0.8\sqrt{15})$$

3.3.2 Solutions (3.2 Exercise)

Answer 1.

(i)	$4\sqrt{5}$	**(ii)**	$4\sqrt{7}$	**(iii)**	$12\sqrt{6}$
(iv)	$4\sqrt{3}$	**(v)**	$\frac{5\sqrt{13}}{13}$	**(vi)**	$7\sqrt{6}$
(vii)	$5\sqrt{11}$	**(viii)**	$\frac{\sqrt{2}}{2}$	**(ix)**	$\sqrt{30}$

Answer 2.

(i)	$\frac{4\sqrt{13}}{3}$	**(ii)**	$\frac{8\sqrt{6}}{5}$	**(iii)**	$\frac{7\sqrt{15}}{45}$
(iv)	$\frac{11\sqrt{3}}{36}$	**(v)**	$\frac{3\sqrt{6}}{7}$	**(vi)**	$\frac{3\sqrt{7}}{7}$
(vii)	$\frac{4\sqrt{11}}{5}$	**(viii)**	$7\sqrt{2}$	**(ix)**	$\frac{2\sqrt{14}}{3}$

Chapter 4

Algebra : Core 1

4.1 FOIL involving surds

Here is a GCSE style question on “expanding the bracketed”;

$$(7 + x)\ (4 + x)$$

Suppose now that x in the last example was $\sqrt{5}$

$$(7 + \sqrt{5})\ (4 + \sqrt{5})$$

Now try this question.
Once done, check your answer with mine, over the page.

$$(3 + \sqrt{7})\ (2 + \sqrt{7})$$

$$(3 + \sqrt{7})(2 + \sqrt{7}) = 6 + 3\sqrt{7} + 2\sqrt{7} + 7$$
$$= 13 + 5\sqrt{7}$$

4.2 Exercise

Expand the brackets giving answers as surds.

(i) $(6 + \sqrt{2})(3 + \sqrt{2})$

(ii) $(5 + \sqrt{13})(4 + \sqrt{13})$

(iii) $(7 + 3\sqrt{2})(5 + \sqrt{2})$

(iv) $(2 + \sqrt{3})(1 + 5\sqrt{3})$

(v) $(4 + 7\sqrt{3})(5 + 2\sqrt{3})$

(vi) $(7+3\sqrt{2})^2$

(vii) $(6+\sqrt{5})(3-2\sqrt{5})$

(viii) $(6-5\sqrt{3})^2$

(ix) $(20+3\sqrt{7})(20-3\sqrt{7})$

(x) $(11+2\sqrt{3})(11-2\sqrt{3})$

4.3 Answers.

4.3.1 Solutions (4.1 Introductory Examples)

$$(7 + x)(4 + x) = 28 + 7x + 4x + x^2$$
$$= 28 + 11x + x^2$$

$$(7 + \sqrt{5})(4 + \sqrt{5}) = 28 + 7\sqrt{5} + 4\sqrt{5} + 5$$
$$= 33 + 11\sqrt{5}$$

4.3.2 Solutions (4.2 Exercise)

(i)

$$(6 + \sqrt{2})(3 + \sqrt{2}) = 18 + 6\sqrt{2} + 3\sqrt{2} + 2$$
$$= 20 + 9\sqrt{2}$$

(ii)

$$(5 + \sqrt{13})(4 + \sqrt{13}) = 20 + 5\sqrt{13} + 4\sqrt{13} + 13$$
$$= 33 + 9\sqrt{13}$$

(iii)

$$(7 + 3\sqrt{2})(5 + \sqrt{2}) = 35 + 7\sqrt{2} + 15\sqrt{2} + 6$$
$$= 41 + 22\sqrt{2}$$

(iv)

$$(2 + \sqrt{3})(1 + 5\sqrt{3}) = 2 + 10\sqrt{3} + \sqrt{3} + 15$$
$$= 17 + 11\sqrt{3}$$

(v)

$$(4 + 7\sqrt{3})(5 + 2\sqrt{3}) = 20 + 8\sqrt{3} + 35\sqrt{3} + 42$$
$$= 62 + 43\sqrt{3}$$

(vi)

$$(7 + 3\sqrt{2})(7 + 3\sqrt{2}) = 49 + 21\sqrt{2} + 21\sqrt{2} + 18$$
$$= 67 + 42\sqrt{2}$$

(vii)

$$(6 + \sqrt{5})\ (3 - 2\sqrt{5}) = 18 - 12\sqrt{5} + 3\sqrt{5} - 10$$
$$= 8 - 9\sqrt{5}$$

(viii)

$$(6 - 5\sqrt{3})\ (6 - 5\sqrt{3}) = 36 - 30\sqrt{3} - 30\sqrt{3} + 75$$
$$= 111 - 60\sqrt{3}$$

(ix)

$$(20 + 3\sqrt{7})\ (20 - 3\sqrt{7}) = 400 - 60\sqrt{7} + 60\sqrt{7} - 63$$
$$= 337$$

(x)

$$(11 + 2\sqrt{3})\ (11 - 2\sqrt{3}) = 121 - 22\sqrt{3} + 22\sqrt{3} - 12$$
$$= 109$$

Chapter 5

5.1 Harder "Rationalising the Denominator"

Previously, we tackled these two particularly interesting 'expand the brackets' problems (from exercise 4.2, last two questions);

(a) $(20 + 3\sqrt{7})\ (20 - 3\sqrt{7})$

(b) $(11 + 2\sqrt{3})\ (11 - 2\sqrt{3})$

The answers of 337 and 109 were remarkable in that they did not contain any square roots. This observation is key in many 'rationalising the denominator' questions.

Example:

To rationalise the denominator of

$$\frac{21}{5 + 3\sqrt{2}}$$

Step 1: Expand the brackets, $(5 + 3\sqrt{2})(5 - 3\sqrt{2})$

Step 2: Multiply both numerator and denominator by $5 - 3\sqrt{2}$

$$\frac{21}{(5 + 3\sqrt{2})} \times \frac{(5 - 3\sqrt{2})}{(5 - 3\sqrt{2})}$$

Note the technique of NOT multiplying out the numerator until first cancelling it against the denominator thus avoiding large number mental arithmetic.

5.2 Exercise

Question 1.

Rationalise the denominator of

$$\frac{132}{9 + 4\sqrt{3}}$$

Step 1 : Expand the brackets : $(9 + 4\sqrt{3}) (9 - 4\sqrt{3})$

Step 2: Multiply both numerator and denominator by $9 - 4\sqrt{3}$

$$\frac{132}{(9 + 4\sqrt{3})} \times \frac{(9 - 4\sqrt{3})}{(9 - 4\sqrt{3})}$$

Question 2.

Rationalise the denominator of

$$\frac{60}{7 + 3\sqrt{5}}$$

Step 1: Expand the brackets : $(7 + 3\sqrt{5}) (7 - 3\sqrt{5})$

Step 2: Multiply both numerator and denominator by $7 - 3\sqrt{5}$

Question 3.

Rationalise the denominator, writing your answer in the form $a + b\sqrt{c}$

$$\frac{\sqrt{7}}{8 - 3\sqrt{7}}$$

Question 4.

(i) Expand the brackets : $(7 + 4\sqrt{3})^2$

(ii) Expand the brackets : $(7 + 4\sqrt{3})(7 - 4\sqrt{3})$

(iii) Hence, rationalise the denominator of $\dfrac{7 + 4\sqrt{3}}{7 - 4\sqrt{3}}$

Question 5.

Rationalise the denominator of $\frac{100\sqrt{5}}{7 + 2\sqrt{11}}$

Write your answer in the form $a\sqrt{5} + b\sqrt{55}$

Question 6

Expand the brackets $(a + b\sqrt{c})\ (a - b\sqrt{c})$

NOTE : The significance of this result is that it shows that our 'rationalising the denominator' technique will always work.

Question 7

Rationalise the denominator of $\dfrac{46}{75\sqrt{2} + 3\sqrt{3}}$

Write your answer in the form $a\sqrt{2} + b\sqrt{3}$

5.3 Answers.

5.3.1 Solutions (5.1 Introductory Examples)

Step 1 :

$$(5 + 3\sqrt{2})\ (5 - 3\sqrt{2}) = 25 - 15\sqrt{2} + 15\sqrt{2} - 18$$
$$= 7$$

Step 2:

$$\frac{21}{(5 + 3\sqrt{2})} \times \frac{(5 - 3\sqrt{2})}{(5 - 3\sqrt{2})}$$
$$= \frac{21\ (5 - 3\sqrt{2})}{7}$$
$$= 3\ (5 - 3\sqrt{2})$$
$$= 15 - 9\sqrt{2}$$

5.3.2 Solutions (5.2 Exercise)

Answer 1.

Step 1 :

$$(9 + 4\sqrt{3})\ (9 - 4\sqrt{3}) = 81 - 36\sqrt{3} + 36\sqrt{3} - 48$$
$$= 33$$

Step 2:

$$\frac{132}{(9 + 4\sqrt{3})} \times \frac{(9 - 4\sqrt{3})}{(9 - 4\sqrt{3})}$$
$$= \frac{132\ (9 - 4\sqrt{3})}{33}$$
$$= 4\ (9 - 4\sqrt{3})$$
$$= 36 - 16\sqrt{3}$$

Answer 2.
Step 1 :

$$(7 + 3\sqrt{5})(7 - 3\sqrt{5}) = 49 - 21\sqrt{5} + 21\sqrt{5} - 45$$
$$= 4$$

Step 2:

$$\frac{60}{(7 + 3\sqrt{5})} \times \frac{(7 - 3\sqrt{5})}{(7 - 3\sqrt{5})}$$
$$= \frac{60\,(7 - 3\sqrt{5})}{4}$$
$$= 15\,(7 - 3\sqrt{5})$$
$$= 105 - 45\sqrt{5}$$

Answer 3.
Step 1 :

$$(8 + 3\sqrt{7})(8 - 3\sqrt{7}) = 64 - 24\sqrt{7} + 24\sqrt{7} - 63$$
$$= 1$$

Step 2:

$$\frac{\sqrt{7}}{(8 - 3\sqrt{7})} \times \frac{(8 + 3\sqrt{7})}{(8 + 3\sqrt{7})}$$
$$= \frac{\sqrt{7}\,(8 + 3\sqrt{7})}{1}$$
$$= 21 + 8\sqrt{7}$$

Answer 4.
(i)

$$(7 + 4\sqrt{3})(7 + 4\sqrt{3}) = 49 + 28\sqrt{3} + 28\sqrt{3} + 48$$
$$= 97 + 56\sqrt{3}$$

(ii)

$$(7 + 4\sqrt{3})(7 - 4\sqrt{3}) = 49 - 28\sqrt{3} + 28\sqrt{3} - 48$$
$$= 1$$

(iii)

$$\frac{(7 + 4\sqrt{3})}{(7 - 4\sqrt{3})} \times \frac{(7 + 4\sqrt{3})}{(7 + 4\sqrt{3})}$$
$$= 97 + 56\sqrt{3}$$

Answer 5.

$$\frac{100\sqrt{5}}{(7 + 2\sqrt{11})} \times \frac{(7 - 2\sqrt{11})}{(7 - 2\sqrt{11})}$$

$$= \frac{100\sqrt{5}\ (7 - 2\sqrt{11})}{49 - 14\sqrt{11} + 14\sqrt{11} - 44}$$

$$= 20\sqrt{5}\ (7 - 2\sqrt{11})$$

$$= 140\sqrt{5} - 40\sqrt{55}$$

Answer 6.

$a^2 - b^2 c$ which is an expression that does not contain square roots.

Answer 7

$$\frac{46}{(5\sqrt{2} + 3\sqrt{3})} \times \frac{(5\sqrt{2} - 3\sqrt{3})}{(5\sqrt{2} - 3\sqrt{3})}$$

$$= \frac{46(5\sqrt{2} - 3\sqrt{3})}{50 - 15\sqrt{6} + 15\sqrt{6} - 27}$$

$$= \frac{46(5\sqrt{2} - 3\sqrt{3})}{50 - 27}$$

$$= \frac{46(5\sqrt{2} - 3\sqrt{3})}{23}$$

$$= 2(5\sqrt{2} - 3\sqrt{3})$$

$$= 10\sqrt{2} - 6\sqrt{3})$$

Chapter 6

Algebra : Core 1

6.1 Surd arithmetic work-out (Homework)

Example

Calculate $2\sqrt{15} \times 4\sqrt{10}$

Write your answer in the form $a\sqrt{b}$ where a and b are integers and b is ☐ free.

6.2 Exercise

Question 1.

Calculate each of the following.

Write your answers in the form $a\sqrt{b}$ where a and b are integers and b is ☐ free.

(i) $5\sqrt{6} \times 11\sqrt{21}$

(ii) $5\sqrt{14} \times 3\sqrt{10}$

(iii) $10\sqrt{22} \times 4\sqrt{6}$

(iv) $3\sqrt{10} \times 4\sqrt{55}$

Question 2.
Rationalise the denominator

(i)

$$\frac{3}{2\sqrt{3}}$$

(ii)

$$\frac{3}{4\sqrt{5}}$$

(iii)

$$\frac{21}{\sqrt{7}}$$

(iv)

$$\frac{18}{11\sqrt{3}}$$

Question 3.
Rationalise the denominator

(i)

$$\frac{1}{2 - \sqrt{3}}$$

(ii)

$$\frac{1}{\sqrt{5} + \sqrt{3}}$$

Question 4.

(i) Write 252 as a product of primes.

(ii) If $\sqrt{252} = x\sqrt{7}$ find x.

Question 5.

(i) Write 882 as a product of primes.

(ii) If $\sqrt{882} = y\sqrt{2}$ find y.

Question 6.

Using your answers from questions 4 and 5, or otherwise, find $\sqrt{252} \times \sqrt{882}$.

Question 7.
Rationalise the denominator

(i)

$$\frac{3}{2\sqrt{5} + 1}$$

(ii)

$$\frac{\sqrt{2}}{3\sqrt{2} - 1}$$

Question 8.

Calculate each of the following.

Write your answers in the form $a\sqrt{b}$ where a and b are integers and b is ☐ free.

(i) $3\sqrt{15} \times 7\sqrt{10}$

(ii) $5\sqrt{21} \times \sqrt{45}$

(iii) $2\sqrt{70} \times \sqrt{90}$

(iv) $3\sqrt{175} \times 2\sqrt{245}$

Question 9.

Rationalise the denominator

$$\frac{1 + \sqrt{3}}{2 - \sqrt{3}}$$

Question 10.

(i) Write 726 as a product of primes.

(ii) If $\sqrt{726} = v\sqrt{6}$ find v.

Question 11.

(i) Write 1350 as a product of primes.

(ii) If $\sqrt{1350} = w\sqrt{6}$ find w.

Question 12.

Using your answers from questions 10 and 11, or otherwise, find $\sqrt{726} + \sqrt{1350}$.

NOTE : That's a + and not a ×.

Question 13.
Rationalise the denominator

$$\frac{7\sqrt{2}}{\sqrt{3} + \sqrt{5}}$$

Question 14.
Simplify each of the following;

(i) $\sqrt{2541} + \sqrt{3024}$

(ii) $\sqrt{3146} - \sqrt{936}$

(iii) $\sqrt{10935} - \sqrt{2940}$

Question 15.

Write in the form $a + b\sqrt{c}$ where a, b and c are integers.

(i) $\dfrac{75 + 3\sqrt{6000}}{5}$

(ii) $\dfrac{-64 + \sqrt{384}}{8}$

(iii) $\frac{7+5\sqrt{392}}{7}$

6.3 Answers.

6.3.1 Solutions (6.1 Introductory Example)

$$2\sqrt{15} \times 4\sqrt{10} = 2\sqrt{3}\ \sqrt{5}\ 4\ \sqrt{2}\ \sqrt{5}$$
$$= 40\sqrt{6}$$

6.3.2 Solutions (6.2 Exercise)

Answer 1.

(i) $165\sqrt{14}$ **(ii)** $30\sqrt{35}$

(iii) $80\sqrt{33}$ **(iv)** $60\sqrt{22}$

Answer 2.

(i) $\dfrac{\sqrt{3}}{2}$ **(ii)** $\dfrac{3\sqrt{5}}{20}$

(iii) $3\sqrt{7}$ **(iv)** $\dfrac{6\sqrt{3}}{11}$

Answer 3.

(i) $2 + \sqrt{3}$ **(ii)** $\dfrac{\sqrt{5} - \sqrt{3}}{2}$

Answer 4.

(i) $252 = 2^2\ 3^2\ 7$

(ii) $6\sqrt{7}$ i.e. $x = 6$

Answer 5.

(i) $882 = 2\ 3^2\ 7^2$

(ii) $21\sqrt{2}$ i.e. $y = 21$

Answer 6.

$126\sqrt{14}$

Answer 7.

(i)

$$\frac{6\sqrt{5} - 3}{19}$$

(ii)

$$\frac{6 + \sqrt{2}}{17}$$

Answer 8.

(i) $105\sqrt{6}$

(ii) $15\sqrt{105}$

(iii) $60\sqrt{7}$

(iv) $210\sqrt{35}$

Answer 9

$5 + 3\sqrt{3}$

Answer 10.

(i) $726 = 2 \;\; 3 \;\; 11^2$

(ii) $11\sqrt{6}$ i.e. $v = 11$

Answer 11.

(i) $1350 = 2 \;\; 3^3 \;\; 5^2$

(ii) $15\sqrt{6}$ i.e. $w = 15$

Answer 12.

$26\sqrt{6}$

Answer 13.

$$\frac{7\sqrt{2}\,(\sqrt{5} - \sqrt{3})}{2}$$

$$= \frac{7\sqrt{10} - 7\sqrt{6}}{2}$$

Answer 14.

(i) $2541 = 3 \;\; 7 \;\; 11^2$ $\quad \therefore \sqrt{2541} = 11\sqrt{21}$

$3024 = 2^4 \;\; 3^3 \;\; 7$ $\quad \therefore \sqrt{3024} = 12\sqrt{21}$ $\quad$ answer : $23\sqrt{21}$

(ii) $3146 = 2 \;\; 11^2 \; 13$ $\quad \therefore \sqrt{3146} = 11\sqrt{26}$

$936 = 2^3 \;\; 3^2 \;\; 13$ $\quad \therefore \sqrt{936} = 6\sqrt{26}$ $\quad$ answer : $5\sqrt{26}$

(iii) $10935 = 3^7 \;\; 5$ $\quad \therefore \sqrt{10935} = 27\sqrt{15}$

$2940 = 2^2 \, 3 \;\; 5 \;\; 7^2$ $\quad \therefore \sqrt{2940} = 14\sqrt{15}$ $\quad$ answer : $13\sqrt{15}$

Answer 15.

(i) $15 + 12\sqrt{15}$

(ii) $-8 + \sqrt{6}$ $\quad$ using $384 = 2^7 \, 3$

(iii) $1 + 10\sqrt{2}$ $\quad$ using $392 = 2^3 \, 7^2$

Chapter 7

Algebra : Core 1

7.1 Old exam Questions on Surds & Indices

7.2 Indices Examples

Write down the values of

(i)	**(ii)**	**(iii)**	**(iv)**	**(v)**
$25^{\frac{1}{2}}$	$27^{\frac{1}{3}}$	3^{-2}	$4^{\frac{3}{2}}$	$9^{-\frac{1}{2}}$
(vi)	**(vii)**	**(viii)**	**(ix)**	**(x)**
$100^{-\frac{3}{2}}$	$81^{\frac{1}{4}}$	$81^{\frac{3}{4}}$	$81^{-\frac{1}{2}}$	81^{0}

7.3 Exercise

Question 1.

C1, May 2005, Q1

(a) Write down the value of

$$8^{\frac{1}{3}}$$

[1 mark]

(b) Find the value of

$$8^{-\frac{2}{3}}$$

[2 marks]

Question 2.

C1, May 2007, Q1

Simplify

$$(3 + \sqrt{5})\,(3 - \sqrt{5})$$

[2 marks]

Question 3.

C1, June 2009, Q1

(a) Simplify

$$(3\sqrt{7})^2$$

[1 mark]

(b) Simplify

$$(8 + \sqrt{5})(2 - \sqrt{5})$$

[3 marks]

Question 4.

C1, January 2009, Q1

(a) Write down the value of

$$125^{\frac{1}{3}}$$

[1 mark]

(b) Find the value of

$$125^{-\frac{2}{3}}$$

[2 marks]

Question 5.

C1, January 2009, Q3

Expand and simplify

$$(\sqrt{7} + 2)(\sqrt{7} - 2)$$

[2 marks]

Question 6.
C1, June 2009, Q2
Given that

$$32\sqrt{2} = 2^a$$

find the value of a.

[3 marks]

Question 7.
C1, January 2005, Q1
(a) Write down the value of

$$16^{\frac{1}{2}}$$

[1 mark]

(b) Find the value of

$$16^{-\frac{3}{2}}$$

[2 marks]

Question 8.
C1, January 2007, Q2
(a) Express $\sqrt{108}$ in the form $a\sqrt{3}$, where a is an integer.

[1 mark]

(b) Express $(2 - \sqrt{3})^2$ in the form $b + c\sqrt{3}$, where b and c are integers to be found.

[3 marks]

Question 9.

C1, May 2006, Q6

(a) Expand and simplify

$$\left(4 + \sqrt{3}\right)\left(4 - \sqrt{3}\right)$$

[2 marks]

(b) Express

$$\frac{26}{4 + \sqrt{3}}$$

in the form $a + b\sqrt{3}$, where a and b are integers.

[2 marks]

Question 10.

C1, January 2010, Q2

(a) Expand and simplify

$$\left(7 + \sqrt{5}\right)\left(3 - \sqrt{5}\right)$$

[3 marks]

(b) Express

$$\frac{7 + \sqrt{5}}{3 + \sqrt{5}}$$

in the form $a + b\sqrt{5}$, where a and b are integers.

[3 marks]

Question 11.

C1, May 2010, Q1

Write

$$\sqrt{75} - \sqrt{27}$$

in the form $k\sqrt{x}$, where k and x are integers.

[2 marks]

Question 12.

C1, January 2012, Q2

(a) Simplify

$$\sqrt{32} + \sqrt{18}$$

giving your answer in the form $a\sqrt{2}$, where a is an integer.

[2 marks]

(b) Simplify

$$\frac{\sqrt{32} + \sqrt{18}}{3 + \sqrt{2}}$$

giving your answer in the form $b\sqrt{2} + c$, where b and c are integers.

[4 marks]

Question 13.

C1, January 2006, Q5

(a) Write $\sqrt{45}$ in the form $a\sqrt{5}$, where a is an integer.

[1 mark]

(b) Express $\dfrac{2(3+\sqrt{5})}{(3-\sqrt{5})}$ in the form $b + c\sqrt{5}$, where b and c are integers.

[5 marks]

Question 14.

C1, January 2008, Q2

(a) Write down the value of

$$16^{\frac{1}{4}}$$

[1 mark]

(b) Simplify

$$(16x^{12})^{\frac{3}{4}}$$

[2 marks]

Question 15.

C1, January 2008, Q3

Simplify

$$\frac{5-\sqrt{3}}{2+\sqrt{3}}$$

giving your answer in the form $a + b\sqrt{3}$, where a and b are integers.

[4 marks]

Question 16.

C1, May 2013, Q1

Simplify

$$\frac{7+\sqrt{5}}{\sqrt{5}-1}$$

giving your answer in the form $a + b\sqrt{5}$, where a and b are integers.

[4 marks]

Question 17.

C1, May 2013, Q3

(a) Find the value of

$$8^{\frac{5}{3}}$$

[2 marks]

(b) Simplify fully

$$\frac{\left(2x^{\frac{1}{2}}\right)^3}{4x^2}$$

[3 marks]

Question 18.

C1, May 2014, Q2

(a) Write down the value of

$$32^{\frac{1}{5}}$$

[1 mark]

(b) Simplify fully

$$(32x^5)^{-\frac{2}{5}}$$

[3 marks]

Question 19.

C1, May 2014, Q6

(a) Write $\sqrt{80}$ in the form $c\sqrt{5}$, where c is a positive constant

[1 mark]

A rectangle R has a length of $(1 + \sqrt{5})$ cm and an area of $\sqrt{80}$ cm^2

(b) Calculate the width of R in cm.
Express your answer in the form $p + q\sqrt{5}$, where p and q are integers to be found.

[4 marks]

Question 20.

C1, May 2016, Q2

Express 9^{3x+1} in the form 3^y giving y in the form $ax + b$
where a and b are constants.

[2 marks]

Question 21.
C1, May 2012, Q3

Show that

$$\frac{2}{\sqrt{12} - \sqrt{8}}$$

can be written in the form $\sqrt{a} + \sqrt{b}$ where a and b are integers

[5 marks]

Question 22.
C1, May 2015, Q1

Simplify

(a)

$$(2\sqrt{5})^2$$

[1 mark]

(b)

$$\frac{\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$$

giving your answer in the form $a + \sqrt{b}$, where a and b are integers.

[4 marks]

Question 23.
C1, May 2016, Q3

Simplify

(a)

$$\sqrt{50} - \sqrt{18}$$

giving your answer in the form $a\sqrt{2}$ where a is an integer

[2 marks]

(b) Hence, or otherwise, simplify

$$\frac{12\sqrt{3}}{\sqrt{50} - \sqrt{18}}$$

giving your answer in the form $b\sqrt{c}$,
where b and c are integers and $b \neq 1$.

[3 marks]

Question 24.

P1, January 2002, Q1

Given that

$$2^x = \frac{1}{\sqrt{2}} \qquad \textit{and} \qquad 2^y = 4\sqrt{2}$$

(a) find the exact value of x and the exact value of y

[3 marks]

(b) calculate the exact value of

$$2^{y-x}$$

[2 marks]

7.4 Answers.

7.4.1 Solutions (7.2 Introductory Examples)

(i) 5 **(ii)** 3 **(iii)** $\frac{1}{9}$ **(iv)** 8 **(v)** $\frac{1}{3}$

(vi) $\frac{1}{1000}$ **(vii)** 3 **(viii)** 27 **(ix)** $\frac{1}{9}$ **(x)** 1

7.4.2 Solutions (7.3 Exercise)

Answer 1.
(a) 2 **(b)** $\frac{1}{4}$

Answer 2.
4

Answer 3.
(a) 63 **(b)** $11 - 6\sqrt{5}$

Answer 4.
(a) 5 **(b)** $\frac{1}{25}$

Answer 5.
3

Answer 6.
$\frac{11}{2}$

Answer 7.
(a) 4 **(b)** $\frac{1}{64}$

Answer 8.
(a) $6\sqrt{3}$ **(b)** $7 - 4\sqrt{3}$

Answer 9.
(a) 13 **(b)** $8 - 2\sqrt{3}$

Answer 10.
(a) $16 - 4\sqrt{5}$ **(b)** $4 - \sqrt{5}$

Answer 11.
$2\sqrt{3}$

Answer 12.
(a) $7\sqrt{2}$ **(b)** $3\sqrt{2} - 2$

Answer 13.
(a) $3\sqrt{5}$ **(b)** $7 + 3\sqrt{5}$

Answer 14.
(a) 2 **(b)** $8x^9$

Answer 15.
$13 - 7\sqrt{3}$

Answer 16.
$3 + 2\sqrt{5}$

Answer 17.
(a) 32
(b) $2x^{-½}$

Answer 24.
(a) $x = -\frac{1}{2}$ $y = \frac{5}{2}$
(b) 8

Chapter 8

Algebra : Core 2

8.1 Index Equations (Core 2 topic)

Example 1
Solve the equation

$$3^x = 81$$

The solution to this is 'obviously' 4 but the real question is how could we obtain this via a 'system'. Here is how;

$$3^x = 81$$

Take the logarithm of both sides to get...

$$\ln 3^x = \ln 81$$

Using a 'rule of logarithms' this becomes...

$$x \ln 3 = \ln 81$$

$$x = \frac{\ln 81}{\ln 3}$$

$$x = 4$$

Example 2
Solve the equation

$$9^x - 3^{x+1} + 2 = 0$$

After some thought you may spot one solution to this, but there is a second solution that you are unlikely to guess as it's an irrational number.

The 'trick' is to write the 9 as a 3^2.

$$9^x - 3^{x+1} + 2 = 0$$

$$(3^2)^x - 3^{x+1} + 2 = 0$$

$$3^{2x} - 3^x\, 3^1 + 2 = 0$$

$$(3^x)^2 - 3(3^x) + 2 = 0$$

$$z^2 - 3z + 2 = 0 \qquad by\ letting\ z = 3^x$$

$$(z - 2)(z - 1) = 0$$

$$\therefore\ Either\ z = 2 \qquad or \qquad z = 1$$

$$3^x = 2 \qquad or \qquad 3^x = 1$$

$$\ln 3^x = \ln 2 \qquad or \qquad x = 0 \qquad (\ obviously\ ?\ )$$

$$x = \frac{\ln 2}{\ln 3}$$

$$x = 0.631$$

8.2 Exercise

Question 1.

Solve the equation

$$2^x = 3$$

Give your answer accurate to 3 decimal places.

Question 2.

C2, June 2008, Q4

(a) Find, to 3 significant figures, the value of *x* for which

$$5^x = 7$$

[2 marks]

(b) Solve the equation

$$5^{2x} - 12(5^x) + 35 = 0$$

[4 marks]

Question 3.
Show that the equation

$$25^{x} - 5^{x+1} + 4 = 0$$

can be written in the form

$$(5^{x} - 4)(5^{x} - 1) = 0$$

and hence solve the equation.

Question 4.
Solve the equation

$$4^{2x+2} = 8^{3x-2}$$

Question 5.
Solve the equation

$$9^{x} = 27^{2-x}$$

Question 6.
Solve the equation

$$9^{x} - 3^{x+2} + 20 = 0$$

give your answers correct to 3 decimal places.

Question 7.
Solve the equation

$$3^{x} \times 3^{2x+1} = 9^{x}$$

Question 8.
Solve the equation

$$16^{x} \times 8^{4x+3} = 4^{x}$$

Give your answer as an exact fraction.

Question 9.

C2, January 2007, Q4

Solve the equation

$$5^x = 17$$

giving your answer to 3 significant figures

[3 marks]

Question 10.

Solve for x

$$9^x - 2 \times 3^{x+1} + 8 = 0$$

Question 11.
Solve for x

$$4^x - 6(2^x) + 5 = 0$$

Question 12
C1, May 2015, Q7

Given that

$$y = 2^x$$

(a) express 4^x in term of y

[1 mark]

(b) Hence, or otherwise, solve

$$8(4^x) - 9(2^x) + 1 = 0$$

[4 marks]

Question 13

Show that the equation

$$8^{x} - 2^{x+6} = 0$$

can be written in the form

$$2^{x}(2^{2x} - 64) = 0$$

and hence solve the equation.

8.3 Answers

Answer 1.
1.585 *3 decimal places*

Answer 2.
(a) 1.21 *3 significant figures*
(b) 1, 1.21

Answer 3.
(a)

$$\begin{aligned} 25^x - 5^{x+1} + 4 &= 0 \\ (5^2)^x - 5^1 5^x + 4 &= 0 \\ (5^x)^2 - 5(5^x) + 4 &= 0 \\ z^2 - 5z + 4 &= 0 \\ (z-4)(z-1) &= 0 \\ (5^x - 4)(5^x - 1) &= 0 \end{aligned}$$

(b) 0, 0.861

Answer 4.
2

Answer 5.
1.2

Answer 6.
1.262, 1.465 *3 decimal places*

Answer 7.
- 1

Answer 8.
$-\frac{9}{14}$

Answer 9.
$1.76 \quad 3\,significant\,figures$

Answer 10.
0.631, 1.262

Answer 11.
0, 2.322

Answer 13

$$\begin{aligned} 8^x - 2^{x+6} &= 0 \\ (2^3)^x - 2^x 2^6 &= 0 \\ 2^{3x} - 2^x 64 &= 0 \\ 2^x 2^{2x} - 2^x 64 &= 0 \\ 2^x (2^{2x} - 64) &= 0 \end{aligned}$$

$$either\ \ 2^x = 0 \ \ which\ has\ no\ solutions$$

$$or\ 2^{2x} = 64 \ \ which\ gives\ \ x = 3$$