A-Level Pure Mathematics

T R I G O N O M E T R Y

Trigonometric Identities

Chapter 1

Trigonometry : Core 3

1.1 The Addition Formulae

How Brackets Expand In Trigonometry

$$sin\ (A \pm B) = sin\,A\ cos\,B \pm cos\,A\,sin\,B$$

$$cos\ (A \pm B) = cos\,A\ cos\,B \mp sin\,A\ sin\,B$$

$$tan\ (A \pm B) = \frac{tan\,A \pm tan\,B}{1 \mp tan\,A\ tan\,B}$$

1.2 Example

Prove that,

$cos\,(90° - \theta) = sin\,\theta$

$$\begin{aligned} LHS &= cos\,(90 - \theta) \\ &= cos\,90°\ cos\,\theta + sin\,90°\,sin\,\theta \\ &= 0 \times cos\,\theta + 1 \times sin\,\theta \\ &= sin\,\theta \\ &= RHS \qquad \square \end{aligned}$$

1.3 Exercise

Question 1.

Prove the following identity.

Be sure to start $LHS =$ and conclude with $= RHS$

$$sin\ (180° - \theta) = sin\,\theta$$

Question 2.
Prove the following identity.
Be sure to start $LHS =$ and conclude with $= RHS$

$$sin\ (90° + \theta) = cos\,\theta$$

Question 3.
Prove that

$$sin(A + B) + sin(A - B) = 2\,sin\,A\,cos\,B$$

Question 4.

Prove this identity;

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

Question 5.

Prove,

$$\frac{\sin(A + B)}{\cos A \cos B} = \tan A + \tan B$$

Question 6.

By letting $B = A$ in the addition formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$ prove that;

$$\sin 2A = 2 \sin A \cos A$$

Question 7.

Using a similar technique to that of question 6, prove that;

$$\cos 2A = \cos^2 A - \sin^2 A$$

Question 8.

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

The three formulae, proved in Questions 6, 7 and 8, are called "The Double Angle Formulae". They are important and useful and so need to be memorised.

Question 9.

Given that

$$\sin(x - \alpha) = \cos(x + \alpha)$$

prove that

$$\tan x = 1$$

Question 10.
Explain carefully, giving an example of each, the difference between a trigonometric identity and a trigonometric equation.

Question 11.
Solve the following trigonometric equation for x between 0° and 360°

$$2\,sin\,x \;=\; cos\;(\,x\;+\;60°\,)$$

Question 12.

Solve the following trigonometric equation for x between 0° and 360°

$$\cos\,(x + 45°) = \cos x$$

Question 13.

Solve the following trigonometric equation for x between 0° and 360°

$$\sin\,(x - 30°) = \frac{1}{2}\cos x$$

Question 14.
Solve the following trigonometric equation for *x* between 0° and 360°

$$3\,sin\ (x + 10°) = 4\,cos\ (x - 10°)$$

Question 15.
Prove that,

$$\frac{tan(C + D) - tan\,C}{1 + tan(C + D)\ tan\,C} = tan\,D$$

HINT: Let the $(C + D) = A$ and the single $C = B$
Then use the *tan* addition formula backwards

Chapter 2

2.1 The Double Angle Formulae

In the Core 3 examination you are given The Addition Formulae.
However, you are not given The Double Angle Formulae.

You should memorize The Double Angle Formulae and also how to obtain them from The Addition Formulae.

2.2 The Double Angle Formula for *sin* 2*A*

$$sin(A + B) = sin A \; cos B + cos A \; sin B$$

Let $B = A$

$$sin(A + A) = sin A \; cos A + cos A \; sin A$$

$$sin\, 2A = 2\, sin A\, cos A$$

2.3 The Double Angle Formula for *cos* 2*A*

$$cos(A + B) = cos A \; cos B - sin A \; sin B$$

Let $B = A$

$$cos(A + A) = cos A \; cos A - sin A \; sin A$$

$$cos\, 2A = cos^2 A - sin^2 A$$

2.4 The Double Angle Formula for *tan* 2*A*

$$tan(A + B) = \frac{tan A + tan B}{1 - tan A \; tan B}$$

Let $B = A$

$$tan(A + A) = \frac{tan A + tan A}{1 - tan A \; tan A}$$

$$tan\, 2A = \frac{2\, tan A}{1 - tan^2 A}$$

2.5 An Examination Question from January 2007, & again in January 2009

Prove that;

$$\sin 3A = 3\sin A - 4\sin^3 A$$

2.6 Exercise

Question 1.

By changing the 1 for $\cos^2 A + \sin^2 A$, or otherwise, prove this identity;

$$\frac{\sin 2A}{1 + \cos 2A} = \tan A$$

LHS =

Question 2.

By changing both 1s for $\cos^2 A + \sin^2 A$, or otherwise, prove this identity;

$$\sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}} = \tan A$$

LHS =

Question 3.

By use of a difference of two squares, or otherwise, prove this identity;

$$\frac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$$

LHS =

Question 4.

By use of a difference of two squares, or otherwise, prove this identity;

$$\cos^4 A - \sin^4 A = \cos 2A$$

LHS =

Question 5.
Prove that;

$$\cos 3A = 4\cos^3 A - 3\cos A$$

Question 6.
Prove this identity;

$$\cot A - \tan A = 2 \cot 2A$$

Question 7.
Prove this identity;

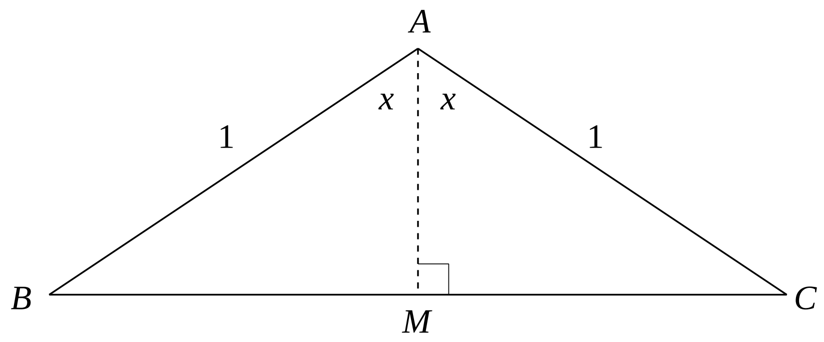
$$\cot A + \tan A = 2 \csc 2A$$

Question 8.

(a) By changing the 1 for $\cos^2 A + \sin^2 A$, prove this identity;

$$1 - 2\sin^2 x = \cos 2x$$

(b)



ABC is an isosceles triangle

$AB = AC = 1$

M is the midpoint of *BC*

(i) Use trigonometry to find an expression, in terms of x, for BM

(ii) Hence write down an expression, in terms of x, for BC

(iii) Use the cosine rule to find an expression, in terms of $\cos 2x$, for BC^2

(iv) Hence show that $\cos 2x = 1 - 2\sin^2 x$

Question 9.

Prove that;

$$tan\ 3A = \frac{3\ tan\ A - tan^3 A}{1 - 3\ tan^2 A}$$

Question 8 (b) based upon an examination question from edexcel, 2008

3.1 Homework

Question 1.

Show, using the formula for $sin\,(A - B)$ that

$$sin\,15° = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Remember to start, LHS =

Question 2.

(a) Write down the exact value of **(i)** $\sin 45°$

(ii) $\cos 45°$

(b) On one graph, sketch the curves of $y = \sin x$ and $y = \cos x$

(c) Use the fact that $\sin 45° = \cos 45°$ to prove that

$$\sin(\theta + 45°) = \cos(\theta - 45°)$$

Question 3.

Prove that;

$$\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} = \frac{\cos\,(A + B)}{\sin B \cos B}$$

Question 4.

Prove that;

$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$$

Question 5.

Prove that,

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$$

Question 6.

Prove that;

$$2\,sin^3\theta\,cos\,\theta\;+\;2\,cos^3\theta\,sin\,\theta\;=\;sin\,2\theta$$

Question 7.

Prove that,

$$\frac{sin\ 3\theta}{sin\ \theta} - \frac{cos\ 3\theta}{cos\ \theta} = 2$$

Chapter 4

Trigonometry : Core 3

4.1 The Reciprocal Trigonometric Functions

The reciprocal trig functions are;

sec $\theta = \dfrac{1}{\cos\theta}$ csc $\theta = \dfrac{1}{\sin\theta}$ cot $\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

We can immediately write down two new useful identities, derived from an old favourite;

$\cos^2\theta + \sin^2\theta \equiv 1$ This is the old favourite

$1 + \tan^2\theta \equiv \sec^2\theta$ Divide through the old favourite by $\cos^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$ Divide through the old favourite by $\sin^2\theta$

The next exercise gives you practice at proving trigonometric identities involving $\sec\theta$, $\csc\theta$ and $\cot\theta$ and, more often than not, the above three identities.

4.1 Exercise

Question 1.

Prove that;

$$\sin\theta\ (\csc\theta - \sin\theta) + \cos\theta\ (\sec\theta - \cos\theta) = 1$$

Question 2.
Prove that;

$$\frac{\sin\theta\ (\sin\theta - \csc\theta)}{\cos\theta\ (\cos\theta - \sec\theta)} = \cot^2\theta$$

Question 3.
Prove that;

$$\frac{(1 - \sin\theta)\ (1 + \sin\theta)}{(1 - \cos\theta)\ (1 + \cos\theta)} = \cot^2\theta$$

Question 4.
Prove that;

$$sec^2\theta - tan^2\theta + csc^2\theta - cot^2\theta = 2$$

Question 5.
Prove that;

$$sec\,\theta\,(cos\,\theta + sin\,\theta\,tan\,\theta) = sec^2\theta$$

Question 6.
Prove that;

$$\frac{1}{csc\,\theta\ (\,cos\,\theta\ cot\,\theta\ +\ sin\,\theta\,)}\ =\ sin^2\,\theta$$

Question 7.
Prove that;

$$\frac{1}{(\,tan^2\,\theta\ +\ 1\,)}\ +\ \frac{1}{(\,cot^2\,\theta\ +\ 1\,)}\ =\ 1$$

Question 8.
Prove that;

$$(\,sec^2\theta - 1\,)\,(\,csc^2\theta - 1\,) = 1$$

Question 9.
Prove that;

$$(\,sec\,\theta + tan\,\theta\,)\,(\,sec\,\theta - tan\,\theta\,) = 1$$

Question 10.

Prove that;

$$\frac{\cos\theta}{\sqrt{1 + \tan^2\theta}} + \frac{\sin\theta}{\sqrt{1 + \cot^2\theta}} = 1$$

Chapter 5

Trigonometry : Core 3

5.1 Past Paper Examination Questions

Question 1.

C3 Examination question from June 2005, Q1.

(a) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta \equiv \sec^2\theta$

[2 marks]

(b) Solve, for $0 \leqslant \theta < 360°$, the equation

$$2\tan^2\theta + \sec\theta = 1$$

giving your answers to 1 decimal place.

[6 marks]

Question 2.

C3 Examination question from January 2013, Q6.

(i) Without using a calculator, find the exact value of

$$(\sin 22.5^\circ + \cos 22.5^\circ)^2$$

You must show each stage of your working.

[5 marks]

(ii) **(a)** Show that $\cos 2\theta + \sin \theta = 1$ may be written in the form

$$k \sin^2 \theta - \sin \theta = 0$$

stating the value of k.

[2 marks]

(b) Hence solve, for $0 \leqslant \theta < 360^\circ$, the equation

$$\cos 2\theta + \sin \theta = 1$$

[4 marks]

Question 3.

C3 Examination question from January 2012, Q5.

Solve, for $0 \leqslant \theta \leqslant 180°$,

$$2\cot^2 3\theta = 7\,\mathrm{cosec}\,3\theta - 5$$

Give your answers in degrees to 1 decimal place.

[10 marks]

Question 4.

C3 Examination question from January 2011, Q3.

Find all the solutions of

$$2\cos 2\theta = 1 - 2\sin\theta$$

in the interval $0 \leqslant \theta < 360°$

[6 marks]

Question 5.

C3 Examination question from June 2011, Q6.

(a) Prove that

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \tan \theta, \qquad \theta \neq 90n°, \; n \in Z$$

[4 marks]

(b) Hence, or otherwise

(i) show that $\tan 15° = 2 - \sqrt{3}$

[3 marks]

(ii) solve, for $0 < x < 360°$,

$$cosec\ 4\theta - \cot 4\theta = 1$$

[5 marks]

Question 6.

C3 Examination question from June 2006, Q6.

(a) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that $\text{cosec}^2\theta - \cot^2\theta \equiv 1$

[2 marks]

(b) Hence, or otherwise, prove that

$$\text{cosec}^4\theta - \cot^4\theta \equiv \text{cosec}^2\theta + \cot^2\theta$$

[2 marks]

(c) Solve, for $90° \leqslant \theta < 180°$,

$$\text{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta$$

[6 marks]

Question 7.

C3 Examination question from June 2007, Q5.

(a) Given that $\sin^2 \theta + \cos^2 \theta \equiv 1$, show that $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$

[2 marks]

(b) Solve, for $0 \leqslant \theta < 180°$, the equation

$$2 \cot^2 \theta - 9 \operatorname{cosec} \theta = 3$$

giving your answers to 1 decimal place.

[6 marks]

Question 8.

C3 Examination question from June 2010, Q1.

(a) Show that

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$$

[2 marks]

(b) Hence find, for $-180° \leqslant \theta \leqslant 180°$, all the solutions of

$$\frac{2 \sin 2\theta}{1 + \cos 2\theta} = 1$$

Give your answers to 1 decimal place.

[3 marks]

Question 9.

C3 Examination question from January 2010, Q8.

Solve

$$cosec^2 2x - cot\, 2x = 1$$

for $0 \leqslant x \leqslant 180°$

[7 marks]

Question 10.

C3 Examination question from January 2007, Q1.

(a) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

[5 marks]

(b) Given that

$$\sin\theta = \frac{\sqrt{3}}{4}$$

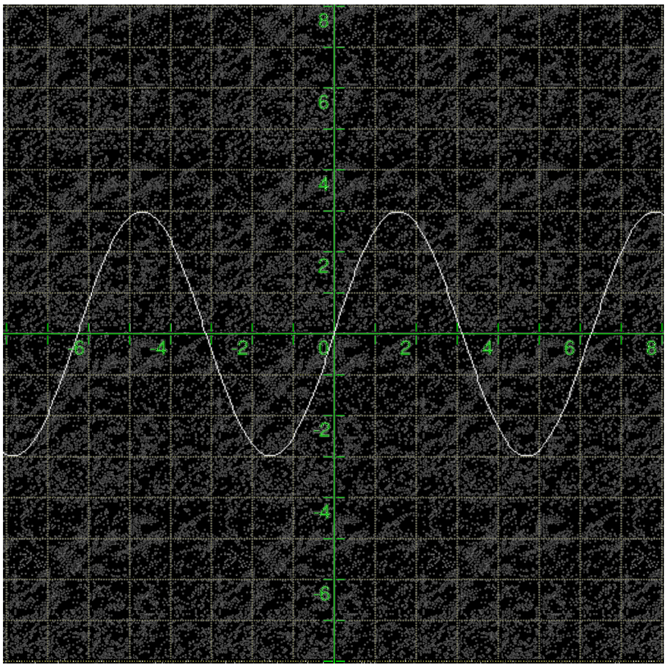
find the exact value of $\sin 3\theta$

[2 marks]

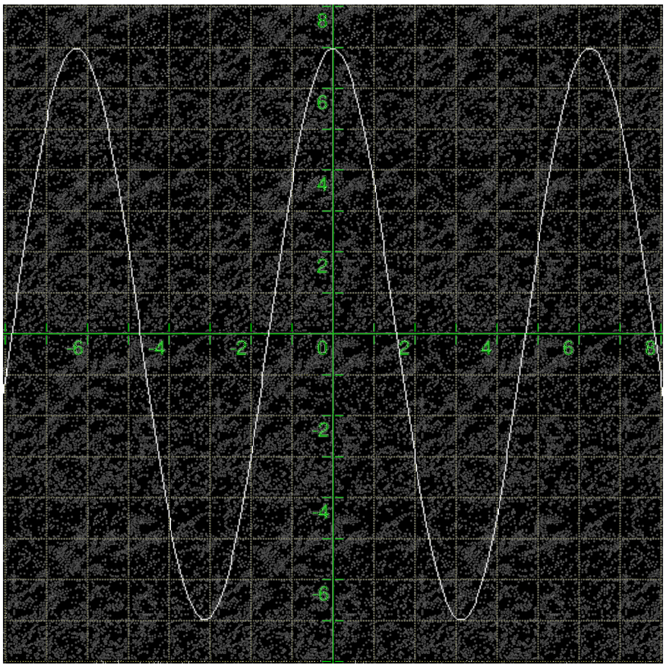
Chapter 6

6.1 Addition of Trigonometric Waveforms

$y = 3\, sin\, \theta$



$y = 7\, cos\, \theta$

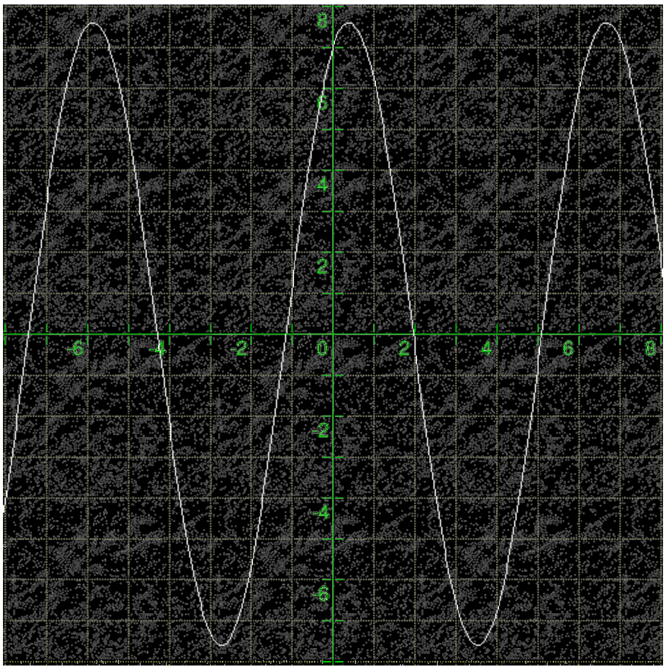


The surprise is about what the following combination curve looks like...

$$y = 3\, sin\, \theta + 7\, cos\, \theta$$

.... how complicated a waveform will this be ?

$$y = 3\,sin\,\theta + 7\,cos\,\theta$$



It's not a complicated wave form at all !
It's a sine wave moved left about 65° (1 and a bit radians) and height between 7 and 8.

Knowing that the resulting wave is a **sine** wave is the key to getting an exact answer because the form of the answer has to be,

$$R\,sin\,(\,\theta + \alpha\,)$$

where α is the shift left, and R is the height or *amplitude*.

Question: Express 3 *sin* θ + 7 *cos* θ in the form R *sin* ($\theta + \alpha$)

Answer:

$$R\,sin\,(\,\theta + \alpha\,) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$$

$$Which\ we\ want\ to = 3\,sin\,\theta + 7\,cos\,\theta$$

$$\therefore\ R\,cos\,\alpha = 3 \quad and \quad R\,sin\,\alpha = 7$$

Solving these simultaneously by division...

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{7}{3}$$

$$\alpha = tan^{-1}\left(\frac{7}{3}\right)$$

$$\alpha = 66.8° \qquad which\ is\ 1.17\ radians$$

$$R = \sqrt{7^2 + 3^2}$$

$$= 7.62$$

Thus : 3 *sin* θ + 7 *cos* θ = 7.62 *sin* (θ + 66.8°)

6.2 Why applying Pythagoras' theorem gives the value of R

$$\sqrt{(R\sin\alpha)^2 + (R\cos\alpha)^2} = \sqrt{R^2\sin^2\alpha + R^2\cos^2\alpha}$$
$$= \sqrt{R^2(\sin^2\alpha + \cos^2\alpha)}$$
$$= \sqrt{R^2} \qquad because \quad \cos^2\alpha + \sin^2\alpha = 1$$
$$= R$$

6.3 Example

(i) Write $8\sin\theta + 15\cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

(ii) What is the minimum value of $8\sin\theta + 15\cos\theta$?

(iii) What is the maximum value of $24\sin\theta + 45\cos\theta$?

(iv) Solve the equation;

$$8\sin\theta + 15\cos\theta = \frac{34}{5} \qquad for\ 0° \leqslant \theta \leqslant 360°$$

6.4 Exercise

Question 1.

(i) Write $2 \sin \theta + \sqrt{5} \cos \theta$ in the form $R \sin (\theta + \alpha)$ for $0 < \alpha < 90°$

(ii) What is the minimum value of $2 \sin \theta + \sqrt{5} \cos \theta$?

(iii) What is the maximum value of $2 \sqrt{5} \sin \theta + 5 \cos \theta$?

(iv) When will $2 \sin \theta + \sqrt{5} \cos \theta = 1.5$?
Give both solutions that are in the interval $0° < \theta < 360°$

Question 2.

(i) Write $\sqrt{3}\,\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

(ii) What is the minimum value of $\sqrt{3}\,\sin\theta + \cos\theta$?

(iii) What is the maximum value of $3\sin\theta + \sqrt{3}\,\cos\theta$?

(iv) When will $\sqrt{3}\,\sin\theta + \cos\theta = \sqrt{2}$?

Give both solutions that are in the interval $0° < \theta < 360°$

The appearance of π in the next question is 'telling you' to give your answers in radians. You may find it easiest to work the question through in degrees, and convert your answers into radians by multiplying by $\frac{\pi}{180}$.

Question 3.

(i) Write $\sin\theta + \sqrt{3}\cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < \frac{\pi}{2}^{c}$

(ii) What is the minimum value of $\sin\theta + \sqrt{3}\cos\theta$?

(iii) What is the maximum value of $\sqrt{5}\sin\theta + \sqrt{15}\cos\theta$?

(iv) Show that $\sin\theta + \sqrt{3}\cos\theta = 1$ has a solution $\frac{11\pi}{6}$ in the interval $0^{c} < \theta < 2\pi^{c}$ and find the other solution in the same interval.

Question 4.

(i) Write $\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < \frac{\pi}{2}^{c}$

(ii) What is the minimum value of $\sin\theta + \cos\theta$?

(iii) What is the maximum value of $\dfrac{1}{\sin\theta + \cos\theta + 3}$?

(iv) Solve the equation $\sin\theta + \cos\theta = 1$ over the interval $0^{c} \leqslant \theta \leqslant 2\pi^{c}$ giving all solutions as exact values.

Question 5.

Find a formula for α in terms of A and B when $A \sin \theta + B \cos \theta$ is written in the form $R \sin(\theta + \alpha)$ for $0 < \alpha < 90°$.

Question 6.

(i) Write down the formula that expands the brackets in $\cos(\theta + \alpha)$

(ii) Write $9 \cos \theta - 12 \sin \theta$ in the form $R \cos(\theta + \alpha)$ for $0 < \alpha < 90°$
Give α accurate to three significant figures.

(iii) Solve the equation $9 \cos \theta - 12 \sin \theta = 0$ for $0 < \alpha < 360°$

Chapter 7

Trigonometry : Core 3

7.1 Revision

Question 1.

(i) Find the four solutions to the following equation,

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad for \quad 0° \leqslant \theta \leqslant 720°$$

[4 marks]

(ii) Hence, or otherwise, solve,

$$\sin(2\theta + 10°) = \frac{\sqrt{3}}{2} \quad , \qquad 0° \leqslant \theta \leqslant 360°$$

[2 marks]

Question 2.

Prove the identity,

$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) + 1 \equiv 2\cos^2\theta$$

[5 marks]

Question 3.
Use the identity,

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

to show that,

$$\tan 75° \equiv \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Rationalise the denominator to further show that for some integer values of *a* and *b*,

$$\tan 75° \equiv a + b\sqrt{3}$$

Clearly state the value of *a* and *b*.

[6 marks]

Question 4.
Prove the identity

$$\csc^2\theta + \sec^2\theta = \csc^2\theta \sec^2\theta$$

[5 marks]

Question 5.

(i) Express $5 \sin x + 12 \cos x$ in the form $R \sin (x + \alpha)$,
where $R > 0$ and α is the smallest positive angle possible.
In your answer express α in degrees and accurate to one decimal place.

(ii) Hence, or otherwise, solve the following equation over $0° \leqslant x \leqslant 360°$

$$5 \sin x + 12 \cos x = 9$$

[8 marks]

Question 6.

(i) By expressing 3θ in the form $2\theta + \theta$, prove that,

$$cos\,3\theta \;=\; 4\,cos^{3}\,\theta \;-\; 3\,cos\,\theta$$

(ii) Hence, or otherwise, solve the equation,

$$cos\,3\theta \;+\; cos\,\theta \;=\; 0 \quad for \quad 0° \leqslant \theta \leqslant 360°$$

[10 marks]

Chapter 8

Trigonometry : Core 3

8.1 TEST

Question 1.

(i) Find the four solutions to the following equation,

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad for \quad 0° \leqslant \theta \leqslant 720°$$

[4 marks]

(ii) Hence, or otherwise, solve,

$$\cos(2\theta + 15°) = \frac{\sqrt{3}}{2} \quad , \qquad 0° \leqslant \theta \leqslant 360°$$

[2 marks]

Question 2.

Prove the identity,

$$4\sin^2\theta\ \cos^2\theta + \cos^2 2\theta \equiv 1$$

[5 marks]

Question 3.
Use the identity,

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

to show that,

$$\tan 15° \equiv \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

Rationalise the denominator to further show that for some integer values of *a* and *b*,

$$\tan 15° \equiv a + b\sqrt{3}$$

Clearly state the value of *a* and *b*.

[6 marks]

Question 4.
Prove the identity

$$\frac{\sin\theta + \cos\theta}{\sin\theta} + \frac{\sin\theta - \cos\theta}{\cos\theta} = \sec\theta \csc\theta$$

[5 marks]

Question 5.

(i) Express $8 \sin x + 15 \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and α is the smallest positive angle possible. Give your answer in degrees and accurate to 1 decimal place.

(ii) Hence, or otherwise, solve the following equation over $0° \leqslant x \leqslant 360°$

$$8 \sin x + 15 \cos x = 13$$

[8 marks]

Question 6.

(i) By expressing 3θ in the form $2\theta + \theta$, prove that,

$$sin\,3\theta \;=\; 3\,sin\,\theta \;-\; 4\,sin^3\theta$$

(ii) Hence, or otherwise, solve the equation,

$$sin\,3\theta \;-\; sin\,\theta \;=\; 0 \qquad for \;\; 0° \leqslant \theta \leqslant 360°$$

[10 marks]